Joining Copulas of Extreme Implicit Dependence Copulas

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June 17, 2025

Abstract

Copulas of uniform-$(0,1)$ random variables U and V satisfying $\alpha(U) = \beta(V)$ almost surely for some measure-preserving transformations α and β are called *implicit dependence copulas.* They were recently shown to coincide with the generalized Markov products of $C_{e,\alpha}$ and $C_{\beta,e}$ with respect to a class of joining copulas $(A_t)_{t\in[0,1]}$. If $C_{e,\alpha}$ and $C_{\beta,e}$ are not two-sided invertible, then most implicit dependence copulas, especially when $A_t \equiv \Pi$, are not extreme points in the class of copulas. For a given pair of left and right invertible copulas $C_{e,\alpha}$ and $C_{\beta,e}$, we characterize extreme implicit dependence copulas in terms of the extremality of the joining copulas in the class of subcopulas on a domain involving the invertible copulas. This result is then extended to the multivariate case.

2020 MSC: Primary 60A10; Secondary 37A30, 47A68, 28A35.
Keywords: generalized Markov product, implicit dependence, complete dependence, extreme copula, joining copula

1 Introduction

Two uniform-$(0,1)$ random variables U and V are said to be *implicitly dependent* if $\alpha(U) = \beta(V)$ almost surely for some measure-preserving transformations α and β on $\mathbb{I}$. The copula of implicitly dependent random variables is called an *implicit dependence copula*, or IDC. For reasons unbeknownst to the authors, the literature on implicit dependence is very limited and we could not find any systematic study of the topic. To the best of our knowledge, implicit dependence copulas first appeared in the literature in the context of measures of dependence in [27]. They introduced a copula-based measure of dependence ω_* in the sense of Rényi [25], of which the first example is Gebelein's maximal correlation [11, 26]. An interesting result is that if the copula of X and Y is a factorizable copula—a

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specific type of an implicit dependence copula—then $\omega_*(X,Y)$ attains its maximum value of 1. In light of this, a potential relationship may exist between Rényi's measures of dependence and implicit dependence copulas, particularly factorizable copulas.

Recently, it was shown in [1, 21] that the implicit dependence copulas correspond exactly to the generalized Markov products of left and right invertible copulas. The *generalized Markov product* of C and D in the class $\mathscr{C}$ of copulas with respect to a parametric class of joining copulas $\mathcal{A} := (A_t)_{t\in\mathbb{I}} \subset \mathscr{C}$ is defined formally by

$$C *_{\mathcal{A}} D(x,y) := \int_0^1 A_t(\partial_2 C(x,t), \partial_1 D(t,y))dt \quad \text{for } x,y \in [0,1]. \tag{1.1}$$

It has a multidimensional version that assembles a d-copula from d bivariate copulas [1]. Evidently [1, 28], $(A_t)_{t\in\mathbb{I}}$ needs to satisfy certain measurability conditions in order for the product (1.1) to be well-defined.

Determining the Fréchet class [9] of multivariate distributions with given marginals, in particular whether the class contains an element, was one of the most important problems in multivariate modeling. By Sklar's theorem [8, 20], the answer to the case when all marginals are univariate is affirmative and given by copulas. However, if some prescribed marginals are multivariate, the problem becomes much more difficult and even the Fréchet class of trivariate distributions with three given bivariate marginals might be empty [7, 29, 30]. When two bivariate marginals of three variables are given, a compatibility condition on the other bivariate marginal can be obtained. In particular, if U_1, U_2, U_3 are uniformly distributed on $[0,1]$ where $(U_1, U_2) \sim C$ and $(U_2, U_3) \sim D$, then the copula of U_1 and U_3 is of the form $C *_{\mathcal{A}} D$ for some $\mathcal{A}$ [7]. In addition, if U_1 and U_3 are functions of the common factor U_2, then each joining copula A_t is the copula of the conditional random variables $U_1 \mid U_2 = t$ and $U_3 \mid U_2 = t$ [1]. In such a case, the complete dependence copulas C and D are $C_{e,\alpha}$ and $C_{\beta,e}$, respectively, and the copula of U_1, U_3 is $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$. Here, e denotes the identity map. And for f, g in the class $\mathscr{T}$ of measure-preserving transformations on $[0,1]$ defined in page 4, the function

$$C_{f,g}(x,y) := \lambda\Big(f^{-1}([0,x]) \cap g^{-1}([0,y])\Big) \tag{1.2}$$

defines a copula and every copula is of this form [4, 34]. When every A_t is the independence copula Π, the generalized Markov product is the usual *Markov product* $*$ possessing nice algebraic properties, e.g., $(\mathscr{C}, *)$ is a monoid with the identity M, the upper Fréchet-Hoeffding bound, and the null element Π. The right and left invertible copulas are exactly the complete dependence copulas $C_{f,e}$ and $C_{e,g}$, respectively. Moreover, M is also the identity for generalized Markov product, i.e., $C *_{\mathcal{A}} M = M *_{\mathcal{A}} C = C$. In particular, $C_{e,f} *_{\mathcal{A}} C_{g,e}$ is a complete dependence copula if either f or g is the identity. Since $C_{f,g} = C_{f,e} * C_{e,g} = C_{f,e} *_{\mathcal{A}} C_{e,g}$ for all f, g and $\mathcal{A}$ [28], the (generalized) Markov products of right invertible and left invertible copulas generate all bivariate copulas. On the other hand, the generalized Markov products of left invertible and right invertible copulas exhibit quite a stringent property, depending on the two copulas. For each pair of $\alpha, \beta \in \mathscr{T}$, different mixtures of copulas in $\mathcal{A}$ give a variety of generalized Markov products $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ supported on the same set as shown in Figure 1. See [6, 7, 8, 28, 32] for additional studies on generalized Markov products.

The class of complete dependence copulas is dense in $\mathscr{C}$ with respect to the uniform metric d_∞ [18] but it is not dense under stronger topologies such as the ones induced by the modified Sobolev norm or the ∂-convergence. As it contains all *checkmin* copulas, the

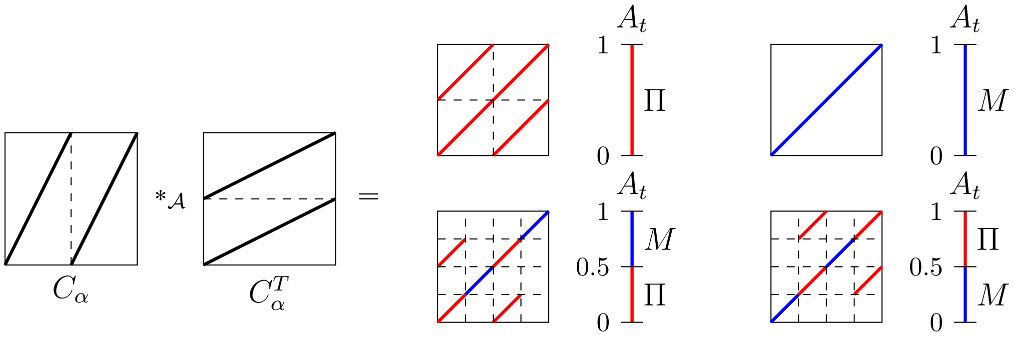

Figure 1: Generalized Markov product $C_\alpha *_{\mathcal{A}} C_\alpha^T$ with different choices of $\mathcal{A} = (A_t)_{t\in\mathbb{I}}$

class of IDCs, or equivalently the class of $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$, is certainly dense in $\mathscr{C}$ under these strong topologies [19, 21]. As such, IDCs could be a tool in investigations of properties which are continuous in a strong topology. However, the IDCs could form a class so large that one might want to investigate the class of IDCs which are extreme elements of $\mathscr{C}$ and its convex hull instead.

For fixed $\alpha, \beta \in \mathscr{T}$, let us denote the class of generalized Markov product of $C_{e,\alpha}$ and $C_{\beta,e}$ by $\mathscr{C}_{\alpha,\beta}$. So the class of IDCs is $\mathscr{C}_{\text{ID}} = \bigcup_{\alpha,\beta\in\mathscr{T}} \mathscr{C}_{\alpha,\beta}$. Since $\mathscr{C}_{\alpha,\beta}$ is convex and closed [1] with respect to d_∞, the class of extreme points of $\mathscr{C}_{\alpha,\beta}$, denoted by $\text{ext}\,(\mathscr{C}_{\alpha,\beta})$, is always nonempty. Clearly, extreme points of $\mathscr{C}$ that are in $\mathscr{C}_{\alpha,\beta}$ are extreme points of $\mathscr{C}_{\alpha,\beta}$. From the fact that $\mu_C = k\mu_A + (1-k)\mu_B$ implies $\mu_A \ll \mu_C$ and $\mu_B \ll \mu_C$, it is also the case that $\text{ext}\,(\mathscr{C}_{\alpha,\beta}) \subseteq \text{ext}(\mathscr{C})$. Hence, $\text{ext}(\mathscr{C}) \cap \mathscr{C}_{\text{ID}} = \bigcup_{\alpha,\beta\in\mathscr{T}} \text{ext}\,(\mathscr{C}_{\alpha,\beta})$ and extreme points of $\mathscr{C}$ in $\mathscr{C}_{\alpha,\beta}$ are identical to extreme points of $\mathscr{C}_{\alpha,\beta}$. In [32, Theorem 9], the extremality of the Markov product of $C_{e,\alpha}$ and $C_{\beta,e}$, also called a *factorizable copula*, is characterized as follows.

Theorem ([32])**.** *For $\alpha, \beta \in \mathscr{T}$, $C_{e,\alpha} * C_{\beta,e}$ is extreme if and only if $\lambda\,(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) = 1$.*

Here, $\mathbb{I}_\gamma$ denotes the injective part of γ defined in Definition 11. Under the condition $\lambda\,(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) = 1$, all generalized Markov products of $C_{e,\alpha}$ and $C_{\beta,e}$ coincide with their Markov product [32, Lemma 8]. Consequently, $\mathscr{C}_{\alpha,\beta}$ is a singleton if $\lambda\,(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) = 1$. On the other hand, when $\lambda\,(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) < 1$, the class $\mathscr{C}_{\alpha,\beta}$ is infinitely larger and $C_{e,\alpha} * C_{\beta,e}$ is no longer extreme. Of course, some elements in $\mathscr{C}_{\alpha,\beta}$ are extreme. See Figure 2 for an example. It should be noted that the smaller the union $\mathbb{I}_\alpha \cup \mathbb{I}_\beta$ or the farther from injectivity the functions α and β, the larger the class $\mathscr{C}_{\alpha,\beta}$. However, all generalized Markov products of $C_{e,\alpha}$ and $C_{\beta,e}$ have mass concentrated on the graph of implicit relation $\alpha(x) = \beta(y)$ [1, 21].

In this manuscript, we expand our investigation to extreme implicit dependence copulas $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ by exploring relationships between the joining copulas (A_t) and the measure-preserving transformations α, β. The contents of this manuscript are organized as follows. All necessary background will be given in the next section. In Section 3, we obtain a sufficient condition as well as a characterization of extreme implicit dependence copulas (Theorems 4 and 7). Section 4 starts off giving a set of sufficient conditions for extreme implicit dependence copulas in Theorem 15 by combining conditions given in Section 3 and the main statement in [32]. Our main result is a complete characterization of extreme implicit dependence copulas given in Theorem 19. The characterizing condition illustrates how the joining copulas and the two factors must be compatible to ensure extremality. Finally, the main result is extended to the multivariate case in Section 5.

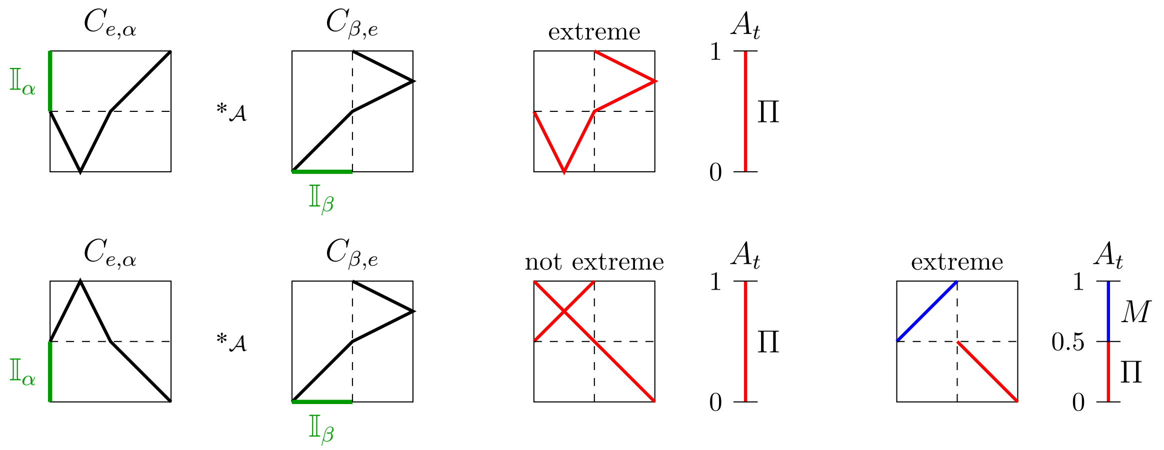

Figure 2: Example of IDCs $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ when (top) $\lambda(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) = 1$ and (bottom) $\lambda(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) < 1$

2 Background

Denote $\mathbb{I} := [0,1]$, $\mathscr{B}$ $\big(\text{resp.}, \mathscr{B}(\mathbb{I}^2)\big)$:= the σ-algebra of Borel subsets of $\mathbb{I}$ (resp., $\mathbb{I}^2$), $\lambda :=$ Lebesgue measure on $(\mathbb{I}, \mathscr{B})$ and $L^1(\mathbb{I})$:= the space of λ-integrable functions on $\mathbb{I}$. Let S_1, S_2 be (closed) subsets of $\mathbb{I}$ containing 0 and 1. A (bivariate) *subcopula* (with domain $S_1 \times S_2$) is a function $C\colon S_1 \times S_2 \to \mathbb{I}$ such that for all $x \in S_1$, $y \in S_2$, $C(0,y) = 0 = C(x,0)$ and $C(1,y) = y$, $C(x,1) = x$, i.e., C is grounded and has uniform marginals; and for every rectangle $B := [x_1,x_2] \times [y_1,y_2]$ whose vertices belong to $S_1 \times S_2$,

$$V_C(B) := C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geqslant 0,$$

i.e., C is 2-increasing. In particular, if S_1 and S_2 are countable sets, then C is also called a *discrete copula* [8, 22, 24]. And C is said to be a (bivariate) *copula* whenever $\mathrm{dom}(C) = \mathbb{I}^2$. Some important copulas are $\Pi\colon (x,y) \mapsto xy$, $M\colon (x,y) \mapsto \min\{x,y\}$ and $W\colon (x,y) \mapsto \max\{x+y-1,0\}$, called the independence, comonotonic, and countermonotonic copulas, respectively. Note that the class of copulas, denoted by $\mathscr{C}$, has a one-to-one correspondence with the class of *doubly stochastic measures*, defined as Borel measures μ on $\big(\mathbb{I}^2, \mathscr{B}(\mathbb{I}^2)\big)$ satisfying $\mu(A \times \mathbb{I}) = \mu(\mathbb{I} \times A) = \lambda(A)$ for all $A \in \mathscr{B}$. The definition of subcopulas also implies that every subcopula is non-decreasing in each variable and Lipschitz with respect to the ℓ^1-norm on $S_1 \times S_2$. Consequently, the first partial derivatives of every copula exist almost everywhere with values in $[0,1]$. See [8, 20] for other basic properties of subcopulas and copulas.

Let $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ be probability spaces. A *measure-preserving* transformation (from X to Y) is a measurable function $f\colon X \to Y$ satisfying $\mu(f^{-1}(B)) = \nu(B)$ for every $B \in \mathscr{Y}$. The class of such maps is denoted by $\mathscr{T}(X,Y)$. In particular, we let $\mathscr{T} := \mathscr{T}(\mathbb{I},\mathbb{I})$ when $(X,\mathscr{X},\mu) = (Y,\mathscr{Y},\nu) = (\mathbb{I},\mathscr{B},\lambda)$. In this manuscript, X and Y are usually Borel sets in $\mathbb{I}$; $\mathscr{X}$ and $\mathscr{Y}$ are the σ-algebras of all Borel subsets of X and Y, respectively; and μ and ν are Lebesgue measures restricted to the corresponding σ-algebras.

Continuous random variables X and Y with $X \sim F$ and $Y \sim G$ are said to be *implicitly dependent* if there exist Borel functions f and g such that $f(X) = g(Y)$ a.s. and $f \circ F^-, g \circ G^- \in \mathscr{T}$, where $A^-(u) := \inf\{x \in \mathbb{R} : A(x) \geqslant u\}$ is the quantile function of A. X and Y are *completely dependent* if Y is a Borel function of X or vice versa. Whenever

X and Y are implicitly/completely dependent continuous random variables, their copula $C_{(X,Y)}$ is called *an implicit/complete dependence copula*. The classes of such copulas are denoted by $\mathscr{C}_{\mathrm{ID}}$ and $\mathscr{C}_{\mathrm{CD}}$, respectively.

Let $\mathscr{D}$ be a convex space, i.e., a subset of a vector space for which $\alpha A + (1-\alpha)\, B \in \mathscr{D}$ for every $A, B \in \mathscr{D}$ and $\alpha \in [0,1]$. Examples of convex spaces are $\mathscr{C}$ and the class of $n \times n$ doubly stochastic matrices for fixed n. A non-negative matrix is *doubly stochastic* if every row and column sums to 1, making it a discrete version of copulas. An element $E \in \mathscr{D}$ is said to be *extreme* if E cannot be written as a strict convex sum of two distinct elements of $\mathscr{D}$, i.e., whenever $E = \alpha A + (1-\alpha) B$ for some $A, B \in \mathscr{D}$ and $\alpha \in (0,1)$ then $A = B = E$. By Birkhoff's Theorem, see [13, 17], every $n \times n$ extreme doubly stochastic matrix is a permutation matrix. However, characterizing extreme copulas is not at all simple. Some known facts on extreme copulas are that all extreme copulas are singular [16] and that, for every copula of the form $C_{f,g}$ as defined in (1.2), the extremal property depends only on $f^{-1}(\mathscr{B})$ and $g^{-1}(\mathscr{B})$. That is, if $f_1, g_1 \in \mathscr{T}$ satisfy $f^{-1}(\mathscr{B}) \approx f_1^{-1}(\mathscr{B})$ and $g^{-1}(\mathscr{B}) \approx g_1^{-1}(\mathscr{B})$,[1] then $C_{f,g}$ is extreme if and only if C_{f_1,g_1} is extreme [34]. The extremality of copulas or doubly stochastic measures (DSMs) have been studied for more than half a century. For instance, extreme DSMs are studied and characterized in functional analysis terms in [5, 16]; they are analyzed using the notion of loops and forests borrowed from graph theory in [2, 12]; and the extremal property of factorizable copulas is investigated in [23, 32].

Denote $\mathbb{1} := \mathbb{1}_{\mathbb{I}}$. A *Markov operator* [8] is a linear map $T\colon L^1(\mathbb{I}) \to L^1(\mathbb{I})$ such that T is positive, $T\mathbb{1} = \mathbb{1}$, and $\int_{\mathbb{I}} Tf \,\mathrm{d}\lambda = \int_{\mathbb{I}} f \,\mathrm{d}\lambda$ for all $f \in L^1(\mathbb{I})$. Let $\mathscr{M}$ be the class of Markov operators. Then every $T \in \mathscr{M}$ has unit norm and its adjoint operator T^*, once extended from $L^\infty(\mathbb{I})$ to $L^1(\mathbb{I})$, is also a Markov operator. An important subclass of $\mathscr{M}$ is the class of $T_\psi\colon f \mapsto f \circ \psi$ for $\psi \in \mathscr{T}$. The identity $T_\psi^* \circ T_\psi = T_e$ implies that T_ψ and T_ψ^* are left and right invertible, respectively. Moreover, the class $(\mathscr{M}, \circ)$ and $(\mathscr{C}, *)$ are isomorphic by the mappings $\Psi\colon \mathscr{M} \to \mathscr{C}$ and $\Phi\colon \mathscr{C} \to \mathscr{M}$, where

$$\Psi(T)(x,y) = \int_0^x \left[T\mathbb{1}_{[0,y]}\right](s)\,\mathrm{d}s \quad \text{and} \quad \left[\Phi(C)f\right](x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \partial_2 C(x,t) f(t)\,\mathrm{d}t.$$

In addition, $\Psi(S \circ T) = \Psi(S) * \Psi(T)$ and $\Phi(C * D) = \Phi(C) \circ \Phi(D)$ for $S, T \in \mathscr{M}$ and $C, D \in \mathscr{C}$. Finally, the above correspondence yields that

$$\partial_1 C(x,y) = T_C \mathbb{1}_{[0,y]}(x) \qquad \text{for a.e. } (x,y) \in \mathbb{I}^2 \tag{2.1}$$

for every copula C with corresponding Markov operator $T_C := \Phi(C)$.

3 Simple extreme implicit dependence copulas

For $\alpha, \beta \in \mathscr{T}$, the extremality of implicit dependence copulas $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$, where $\{A_t\colon t \in \mathbb{I}\}$ is a singleton consisting of M, W or Π was investigated in [32, Theorems 17 and 9]. In this section, we explore two sets of conditions under which $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ is extreme. The first, derived in Subsection 3.1, is that $\{A_t\colon t \in \mathbb{I}\} \subseteq \{M, W\}$, which directly generalizes [32, Theorem 17]. From this result stated in Theorem 4, one might draw a premature conclusion that if each A_t is extreme then $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ is extreme. The

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[1] Here, $\mathscr{R} \approx \mathscr{S}$, for sub-$\sigma$-algebras $\mathscr{R}, \mathscr{S} \subseteq \mathscr{B}$, means for each $R \in \mathscr{R}$ there is $S \in \mathscr{S}$ such that $\lambda(R \triangle S) = 0$ and for each $S \in \mathscr{S}$ there is $R \in \mathscr{R}$ such that $\lambda(R \triangle S) = 0$ where $R \triangle S = (R \backslash S) \cup (S \backslash R)$.

second set of conditions in Subsection 3.2 is more complex and involves the extremality of transformation matrices induced by A_t. Since all transformation matrices induced by M or W are always extreme, Theorem 7 yields Theorem 4 provided that α and β are countably piecewise monotonic surjections. Moreover, it provides better insight into how the extremality of each A_t should be assumed in relation to α and β. This leads to a complete characterization of extreme implicit dependence copulas in Section 4.

3.1 Extremality of implicit dependence copulas whose joining copulas are the Fréchet-Hoeffding bounds

Hereafter, for brevity, the product $*_{(A_t)}$ may be written as $*_A$ if $A_t \equiv A$ for almost all t. By [32, Theorems 17], if $A = M$ or W, then every $C_{e,\alpha} *_A C_{\beta,e}$ is an extreme element. In this subsection, we investigate the extremality of copulas obtained from combining these two cases, i.e., the implicit dependence copulas of the form $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ where $A_t = W$ on S and $A_t = M$ on $\mathbb{I}\backslash S$ a.e. for some Borel subset S of $\mathbb{I}$. By the definition of $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$, the doubly stochastic Borel measure μ_C is uniquely defined from its value for rectangles $R := (a,b] \times (c,d] \subseteq \mathbb{I}^2$. In fact, $\mu_C = \mu_{C,\mathbb{I}}$ where

$$\mu_{C,S}(R) = \int_S V_{A_t}\left((\partial_2 C_{e,\alpha}(a,t), \partial_2 C_{e,\alpha}(b,t)] \times (\partial_1 C_{\beta,e}(t,c), \partial_1 C_{\beta,e}(t,d)]\right) \mathrm{d}\lambda(t), \tag{3.1}$$

for $S \in \mathscr{B}$. Furthermore, it satisfies the following properties which are inspired by [1, Theorem 2.7].

Lemma 1. *Let $\alpha, \beta \in \mathscr{T}$, $(A_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}$ and $S \in \mathscr{B}$ with $\lambda(S) > 0$. If $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ then $\mu_{C,S}(\mathrm{Gr}_{\alpha,\beta}(S')) = \lambda(S \cap S')$ where $\mathrm{Gr}_{\alpha,\beta}(B) := \left\{(x,y) \in \mathbb{I}^2 : \alpha(x) = \beta(y) \in B\right\}$.*

Proof. Let $U, V \sim \mathcal{U}(0,1)$ and C the copula of (U,V). Then $Z := \alpha(U) \sim \mathcal{U}(0,1)$ and, by [32, Theorem 6], $\alpha(U) = \beta(V)$ a.s. Moreover, $(U,Z) \sim C_{e,\alpha}$ and $(Z,V) \sim C_{\beta,e}$. Now, $A_t(\partial_2 C_{e,\alpha}(u,t), \partial_1 C_{\beta,e}(t,v))$ can be interpreted as the conditional probability $\mathbb{P}(U \in [0,u], V \in [0,v] \mid \alpha(U) = \beta(V) = t)$ (see [1]). Then for any rectangle $R := (a,b] \times (c,d] \subseteq \mathbb{I}^2$, the integrand of $\mu_{C,S}(R)$ (see (3.1)) equals $\mathbb{P}((U,V) \in R \mid \alpha(U) = \beta(V) = t)$ which implies by the disintegration theorem [15] that

$$\mu_{C,S}(R) = \int_S \mathbb{P}((U,V) \in R \mid \alpha(U) = \beta(V) = t)\, \mathrm{d}\lambda(t) = \mathbb{P}((U,V) \in R \cap \mathrm{Gr}_{\alpha,\beta}(S)). \tag{3.2}$$

Hence, (3.2) holds for any $R \in \mathscr{B}(\mathbb{I}^2)$. In particular, $\mu_{C,S}(\mathrm{Gr}_{\alpha,\beta}(S')) = \mathbb{P}((U,V) \in \mathrm{Gr}_{\alpha,\beta}(S' \cap S)) = \mu_{C,S\cap S'}(\mathbb{I}^2) = \int_{S\cap S'} V_{A_t}(\mathbb{I} \times \mathbb{I})\, \mathrm{d}\lambda(t) = \int_{S\cap S'} \mathrm{d}\lambda(t) = \lambda(S \cap S')$. ☐

As a consequence, μ_C can be split into two measures with essentially disjoint supports. Verifying the extremality of μ_C amounts to proving that of each component separately. The next tool involves the generalized Markov product of left and right invertible copulas. Proved by a simple change of variable, the following lemma says essentially that shuffling the invertible copulas amounts to reordering the joining copulas.

Lemma 2. *Let $\alpha, \beta \in \mathscr{T}$ and $(A_t)_{t\in\mathbb{I}}$ be a collection of joining copulas. Then for $\varphi \in \mathscr{T}_{\mathrm{inv}}$, $C_{e,\varphi\circ\alpha} *_{(A_t)} C_{\varphi\circ\beta,e} = C_{e,\alpha} *_{\left(A_{\varphi(t)}\right)} C_{\beta,e}$. Moreover, if ψ is an essential inverse of φ, then $C_{e,\varphi\circ\alpha} *_{\left(A_{\psi(t)}\right)} C_{\varphi\circ\beta,e} = C_{e,\alpha} *_{(A_t)} C_{\beta,e}$.*

Proof. It clearly suffices to verify only the first statement. For $x, y, t \in \mathbb{I}$,

$$\begin{aligned}\partial_2 C_{e,\varphi\circ\alpha}(x,t) &= \partial_1 C_{\varphi\circ\alpha,e}(t,x) = T^*_{\varphi\circ\alpha}\mathbb{1}_{[0,x]}(t) = (T_\psi \circ T^*_\alpha)\,\mathbb{1}_{[0,x]}(t)\\ &= T^*_\alpha \mathbb{1}_{[0,x]}\,(\psi(t)) = \partial_1 C_{\alpha,e}\,(\psi(t),x) = \partial_2 C_{e,\alpha}\,(x,\psi(t))\end{aligned}$$

and $\partial_1 C_{\varphi\circ\beta,e}(t,y) = \partial_1 C_{\beta,e}\,(\psi(t),y)$. Hence, by a change of variable,

$$\begin{aligned}C_{e,\varphi\circ\alpha} *_{(A_t)} C_{\varphi\circ\beta,e}(x,y) &= \int_0^1 A_t\left(\partial_2 C_{e,\varphi\circ\alpha}(x,t), \partial_1 C_{\varphi\circ\beta,e}(t,y)\right) \mathrm{d}t\\ &= \int_0^1 A_t\left(\partial_2 C_{e,\alpha}\,(x,\psi(t))\,, \partial_1 C_{\beta,e}\,(\psi(t),y)\right) \mathrm{d}t\\ &= \int_0^1 A_{\varphi(t)}\left(\partial_2 C_{e,\alpha}\,(x,t)\,, \partial_1 C_{\beta,e}\,(t,y)\right) \mathrm{d}t\\ &= C_{e,\alpha} *_{\left(A_{\varphi(t)}\right)} C_{\beta,e}(x,y).\end{aligned}$$

□

Several theorems in this manuscript rely on the following key result, whose proof is a consequence of results in [1, 21].

Lemma 3. *If a doubly stochastic measure μ_{C_1} is absolutely continuous with respect to μ_C, written $\mu_{C_1} \ll \mu_C$, for some implicit dependence copula $C = C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ then C_1 is necessarily an implicit dependence copula, i.e., there is $\mathcal{A}' := \{A'_t\}_{t\in\mathbb{I}} \subseteq \mathscr{C}$ such that $C_1 = C_{e,\alpha} *_{\mathcal{A}'} C_{\beta,e}$. Recall that $\mu_{C_1} \ll \mu_C$ means $\mu_{C_1}(R) = 0$ whenever $\mu_C(R) = 0$ for all $R \in \mathscr{B}(\mathbb{I}^2)$.*

Proof. From [1, Theorem 2.7] or [21, Theorem 5], $C = C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ is an implicit dependence copula of some pair of standard uniform random variables concentrated on the graph $\alpha(x) = \beta(y)$. As a consequence, $\mu_C\left(\mathrm{Gr}_{\alpha,\beta}(\mathbb{I})\right) = 1$. It then follows from the assumption that $\mu_{C_1}\left(\mathrm{Gr}_{\alpha,\beta}(\mathbb{I})\right) = 1$. Thus by [1, Theorem 2.7] or [21, Theorem 10], there exists a parametric family $\mathcal{A}' \subseteq \mathscr{C}$ such that $C_1 = C_{e,\alpha} *_{\mathcal{A}'} C_{\beta,e}$. □

We are now ready to verify the extremality of implicit dependence copulas in a more general form than those studied in [32, Theorem 17].

Theorem 4. *Let $\alpha, \beta \in \mathscr{T}$, $S \in \mathscr{B}$ and $(A_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}$ be such that $A_t = \begin{cases} W & \text{if } t \in S;\\ M & \text{if } t \in \mathbb{I}\backslash S.\end{cases}$ Then $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ is extreme.*

Proof. In light of [32, Theorem 17] in which the statement is proved for $S = \varnothing$ or $[0,1]$, we suppose that $0 < \lambda(S) < 1$. By the rearrangement theorem [3, Theorem 4.3], there exists $\varphi \in \mathscr{T}_{\mathrm{inv}}$ with essential inverse ψ such that $\psi(S) = [0, \lambda(S))$ and $\psi(\mathbb{I}\backslash S) = [\lambda(S), 1]$. Thus by Lemma 2, $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e} = C_{e,\varphi\circ\alpha} *_{\left(A_{\psi(t)}\right)} C_{\varphi\circ\beta,e} = C_{e,\alpha'} *_{(A_s)} C_{\beta',e}$, where $\alpha' := \varphi \circ \alpha$, $\beta' := \varphi \circ \beta$ and $s := \psi(t)$ which also gives $A_s = W\mathbb{1}_{[0,\lambda(S))}(s) + M\mathbb{1}_{[\lambda(S),1]}(s)$. For concise notation, we let $s = \lambda(S)$ and shall revert to the original setup with $S = [0, s]$.

To show that C is extreme, suppose $\mu_C = k\mu_{C_1} + (1-k)\mu_{C_2}$ for some $C_1, C_2 \in \mathscr{C}$ and $k \in (0,1)$. Then $\mu_{C_1} \ll \mu_C$ and $\mu_{C_2} \ll \mu_C$ which imply by Lemma 3 that $C_1 = C_{e,\alpha} *_{(P_t)} C_{\beta,e}$ and $C_2 = C_{e,\alpha} *_{(Q_t)} C_{\beta,e}$ for some $(P_t)_{t\in\mathbb{I}}, (Q_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}$. Setting $D_1 := C_{e,\alpha} *_W C_{\beta,e}$, $D_2 := C_{e,\alpha} *_M C_{\beta,e}$ and $\mathrm{Gr}_{\alpha,\beta}(B) := \left\{(x,y) \in \mathbb{I}^2 : \alpha(x) = \beta(y) \in B\right\}$ for $B \in \mathscr{B}$, $\mu_{D_1,S}, \mu_{C_1,S}, \mu_{C_2,S}$ become measures with mass concentrated on $\mathrm{Gr}_{\alpha,\beta}(S)$. Likewise, $\mu_{D_2,\mathbb{I}\backslash S}, \mu_{C_1,\mathbb{I}\backslash S}, \mu_{C_2,\mathbb{I}\backslash S}$ are measures supported on $\mathrm{Gr}_{\alpha,\beta}(\mathbb{I}\backslash S)$. In addition,

$$\mu_C = \mu_{D_1,S} + \mu_{D_2,\mathbb{I}\backslash S}, \quad \mu_{C_1} = \mu_{C_1,S} + \mu_{C_1,\mathbb{I}\backslash S} \quad \text{and} \quad \mu_{C_2} = \mu_{C_2,S} + \mu_{C_2,\mathbb{I}\backslash S}.$$

Since $\mathrm{Gr}_{\alpha,\beta}(S)$ and $\mathrm{Gr}_{\alpha,\beta}(\mathbb{I}\backslash S)$ are disjoint, we obtain $\mu_{D_1,S} = k\mu_{C_1,S} + (1-k)\mu_{C_2,S}$ and $\mu_{D_2,\mathbb{I}\backslash S} = k\mu_{C_1,\mathbb{I}\backslash S} + (1-k)\mu_{C_2,\mathbb{I}\backslash S}$. Now, similar to the proof in [32, Theorem 17], the measures $\mu_{D_1,S}$ and $\mu_{D_2,\mathbb{I}\backslash S}$ are extreme, so $\mu_{D_1,S} = \mu_{C_1,S} = \mu_{C_2,S}$ and $\mu_{D_2,\mathbb{I}\backslash S} = \mu_{C_1,\mathbb{I}\backslash S} = \mu_{C_2,\mathbb{I}\backslash S}$ which imply that $\mu_C = \mu_{C_1} = \mu_{C_2}$, i.e., C is extreme as desired. □

Example 5. Let $f, g \in \mathscr{T}$ be defined, as shown in Figure 3, by

$$f(x) = \begin{cases} \frac{3}{2}x & \text{if } 0 \leqslant x \leqslant \frac{1}{3}; \\ x + \frac{1}{6} & \text{if } \frac{1}{3} < x \leqslant \frac{5}{6}; \\ 3 - 3x & \text{if } \frac{5}{6} < x \leqslant 1 \end{cases} \quad \text{and} \quad g(y) = \begin{cases} 1 - 2y & \text{if } 0 \leqslant y \leqslant \frac{1}{4}; \\ \frac{5}{4} - 3y & \text{if } \frac{1}{4} < y \leqslant \frac{5}{12}; \\ \frac{3}{2}y - \frac{5}{8} & \text{if } \frac{5}{12} < y \leqslant \frac{3}{4}; \\ 2y - 1 & \text{if } \frac{3}{4} < y \leqslant 1. \end{cases}$$

Since $\mathbb{I}_f = \left[\frac{1}{2}, 1\right]$ and $\mathbb{I}_g = \varnothing$, we have $\lambda\left(\mathbb{I}_f \cup \mathbb{I}_g\right) = \frac{1}{2} < 1$, and then $C_{e,f} * C_{g,e}$ is not extreme by [32, Theorem 5]. When the usual Markov product is changed into generalized Markov products with joining copulas as a mixture of W and M, we obtain various extreme copulas as shown in Figure 4. However, the support of $C_{e,f} *_{(A_t)} C_{g,e}$ is invariant on $\mathrm{Gr}_{f,g}\left(\left(\frac{1}{2}, 1\right)\right)$ for every collection of joining copulas.

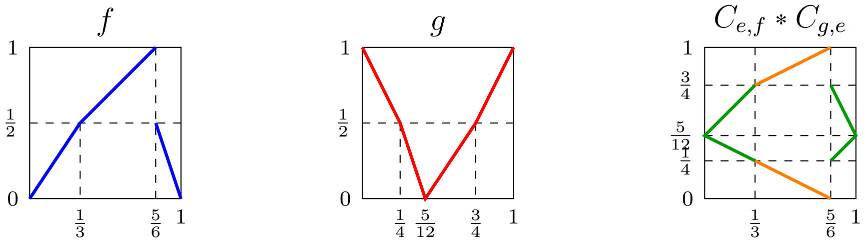

Figure 3: Graphs of functions f and g as well as the support of $C_{e,f} * C_{g,e}$ in Example 5

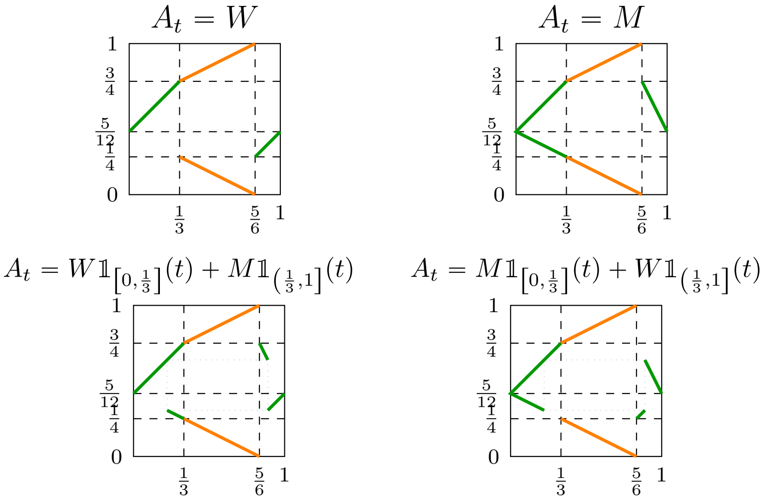

Figure 4: Support of $C_{e,f} *_{(A_t)} C_{g,e}$ for f and g discussed in Example 5 where $(A_t) \subseteq \{W, M\}$

3.2 Extreme countably piecewise monotonic surjection implicit dependence copulas

Countably piecewise monotonic surjection (CPMS) implicit dependence copulas, the class of which is denoted by $\mathscr{C}_{\mathrm{PMID}}$, were first introduced in [21] as copulas of $U, V \sim \mathcal{U}(0,1)$

satisfying $\alpha(U) = \beta(V)$ a.s. for some CPMSs α, β. Here, $\alpha \in \mathscr{T}$ is a CPMS function, written $\alpha \in \mathscr{T}_{\mathrm{CPMS}}$, if there is a countable set $\mathcal{I}$ and a partition (a.e.) $P := \{I_n\}_{n\in\mathcal{I}}$ of $[0,1]$ consisting of open intervals $I_n := (a_n, b_n)$ such that each $\alpha_n := \alpha|_{I_n}$ is a strictly monotonic function from I_n onto $(0,1)$. Recall [3] that $\partial_1 C_{e,\alpha}(x,t) = \mathbb{1}_{[0,t]}(\alpha(x))$ for $\alpha \in \mathscr{T}$. For $\alpha \in \mathscr{T}_{\mathrm{CPMS}}$ with partition $\{I_p := (a_p, b_p)\}_{p\in\mathcal{I}}$, we say that $I_m \prec I_n$ if $b_m \leqslant a_n$ and that $I_m \preceq I_n$ if $I_m \prec I_n$ or $I_m = I_n$. It is shown in [21] that for $x \in I_n$

$$\partial_2 C_{e,\alpha}(x,t) = \mu_n^x(t) := \begin{cases} \mu_n^{\prec}(t) & \text{if } x < \alpha_n^{-1}(t); \\ \mu_n^{\preceq}(t) & \text{if } x > \alpha_n^{-1}(t), \end{cases} \tag{3.3}$$

where $\mu_n^{\prec}(t) := \sum_{I_i \prec I_n} \frac{1}{|\alpha'(\alpha_i^{-1}(t))|}$ and $\mu_n^{\preceq}(t) := \sum_{I_i \preceq I_n} \frac{1}{|\alpha'(\alpha_i^{-1}(t))|}$. Let $\beta \in \mathscr{T}_{\mathrm{CPMS}}$ with partitions $\{J_q\}_{q\in\mathcal{J}}$ and $\eta_q^{\prec}(t)$ and $\eta_q^{\preceq}(t)$ be defined for β in the same way as $\mu_p^{\prec}(t)$ and $\mu_p^{\preceq}(t)$ for α. Clearly, $\{I_p \times J_q\}$ is a rectangular partition (a.e.) of $\mathbb{I}^2$. The equation (3.1) for $(a,b] \subseteq I_p$, $(c,d] \subseteq J_q$ and $S = \mathbb{I}$ becomes

$$\mu_C\left((a,b] \times (c,d]\right) = \int_{[0,1]} V_{A_t}\left(\left(\mu_p^a(t), \mu_p^b(t)\right] \times \left(\eta_q^c(t), \eta_q^d(t)\right]\right) \mathrm{d}\lambda(t).$$

Let us define the notion of transformation matrix with respect to countable partitions, generalizing finite transformation matrices utilized in [10].

Definition 6. Given two countable partitions of $\mathbb{I}$: $P = \{I_p\}_{p\in\mathcal{I}}$ and $Q = \{J_q\}_{q\in\mathcal{J}}$, where I_p and J_q are open intervals, a *transformation* matrix is a non-negative matrix $T = [t_{pq}]_{(p,q)\in\mathcal{I}\times\mathcal{J}}$ for which the sum along each column p is $\lambda(I_p)$ and the sum along each row q is $\lambda(J_q)$, i.e., $\sum_{q\in\mathcal{J}} t_{pq} = \lambda(I_p)$ and $\sum_{p\in\mathcal{I}} t_{pq} = \lambda(J_q)$. Accordingly, P and Q are called the corresponding *horizontal partition* and *vertical partition* of $\mathbb{I}$, respectively. Let $\mathcal{M}(P,Q)$ denote the class of transformation matrices whose horizontal and vertical partitions of $\mathbb{I}$ are P and Q, respectively. Under element-wise operations, $\mathcal{M}(P,Q)$ is clearly convex. Its extreme points shall be studied in relation to the extremality of implicit dependence copulas.

For each $t \in \mathbb{I}$, since $\partial_2 C_{e,\alpha}(\cdot, t)$ and $\partial_1 C_{\beta,e}(t, \cdot)$ can be extended to distribution functions, the collections

$$P(t) := \left\{I'_p(t) := \left(\mu_p^{\prec}(t), \mu_p^{\preceq}(t)\right)\right\}_{p\in\mathcal{I}} \quad \text{and} \quad Q(t) := \left\{J'_q(t) := \left(\eta_q^{\prec}(t), \eta_q^{\preceq}(t)\right)\right\}_{q\in\mathcal{J}} \tag{3.4}$$

are two partitions (a.e.) of $\mathbb{I}$. So $R(t) := \left\{R_{p,q}(t) := I'_p(t) \times J'_q(t)\right\}_{(p,q)\in\mathcal{I}\times\mathcal{J}}$ is a rectangular partition (a.e.) of $\mathbb{I}^2$. Given $(A_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}$, each $t \in \mathbb{I}$ gives a transformation matrix $T(A_t) := [a_{p,q}(t)]_{(p,q)\in\mathcal{I}\times\mathcal{J}}$, where $a_{p,q}(t) := V_{A_t}(R_{p,q}(t))$. Moreover, the value of A_t at pertinent points can be determined from $T(A_t)$ by

$$A_t\left(\mu_p^{\lessdot}(t), \eta_q^{\lhd}(t)\right) = \sum_{I_i \lessdot I_p} \sum_{J_j \lhd J_q} a_{i,j}(t) \tag{3.5}$$

where $\lessdot$ and $\lhd$ are either $\prec$ or $\preceq$. Conversely, for a given transformation matrix $T = [a_{ij}]$ with horizontal and vertical partitions P and Q, respectively, (3.5) can be used to define a subcopula A on $\partial P \times \partial Q$ for which $T(A) = T$. Here, ∂P (resp., ∂Q) is the set of all endpoints of open intervals in P (resp., Q).

Theorem 7. *Let* $\alpha, \beta \in \mathscr{T}_{CPMS}$ *with countable partitions* $\{I_p\}_{p\in\mathcal{I}}$ *and* $\{J_q\}_{q\in\mathcal{J}}$*, respectively. and* $(A_t)_{t\in\mathbb{I}}$ *a collection of joining copulas. For each* $t \in \mathbb{I}$*, let* $P(t)$ *and* $Q(t)$ *be defined by* (3.4) *and* $T(A_t) := [V_{A_t}(R_{p,q}(t))]_{(p,q)\in\mathcal{I}\times\mathcal{J}}$*. Then* $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ *is extreme if and only if* $T(A_t)$ *is extreme in* $\mathcal{M}(P(t), Q(t))$ *for Lebesgue-almost every* $t \in \mathbb{I}$*.*

Extremality of $T(A_t)$, for each t, has many equivalent statements, see e.g. [16, 22, 24]. To prove Theorem 7, we need the following results.

Proposition 8. Let $\alpha, \beta \in \mathscr{T}_{\text{CPMS}}$. For $(p, q) \in \mathcal{I} \times \mathcal{J}$, if $(x, y) \in I_p \times J_q$ satisfies $\alpha(x) = \beta(y)$, then there exists a rectangle $S_{x,y}$ in $\mathbb{I}^2$ such that

$$\mu_{C_{e\alpha} *_{(A_t)} C_{\beta e}}(S_{x,y}) = \int_0^{\alpha(x)} V_{A_t}(R_{p,q}(t))\,\mathrm{d}t,$$

where $R_{p,q}(t) = \big(\mu_p^{\prec}(t), \mu_p^{\preceq}(t)\big) \times \big(\eta_q^{\prec}(t), \eta_q^{\preceq}(t)\big)$. In fact, $S_{x,y} = \alpha_p^{-1}([0, \alpha(x)]) \times \beta_q^{-1}([0, \beta(y)])$.

Proof. It follows from the definition that if $S = [a, b] \times [c, d]$, then $\mu_{C_{e\alpha} *_{(A_t)} C_{\beta e}}(S) = \int_0^1 V_{A_t}(S(t))\,\mathrm{d}t$ where $S(t) = [\partial_2 C_{e,\alpha}(a, t), \partial_2 C_{e,\alpha}(b, t)] \times [\partial_1 C_{\beta,e}(t, c), \partial_1 C_{\beta,e}(t, d)]$. Fix $(p, q) \in \mathcal{I} \times \mathcal{J}$ and $x \in I_p$. Let $y \in J_q$ be such that $\alpha(x) = \beta(y)$ and choose $S = P_x \times Q_y$ where

$$P_x = \begin{cases} [a_p, x] & \text{if } \alpha_p \text{ is increasing;} \\ [x, b_p] & \text{if } \alpha_p \text{ is decreasing,} \end{cases} \quad \text{and} \quad Q_y = \begin{cases} [c_q, y] & \text{if } \beta_q \text{ is increasing;} \\ [y, d_q] & \text{if } \beta_q \text{ is decreasing.} \end{cases}$$

If $S = [a_p, x] \times [c_q, y]$, i.e., both α_p and β_q are increasing, then $S(t) = \big[\mu_p^{\prec}(t), \partial_2 C_{e,\alpha}(x, t)\big] \times \big[\eta_q^{\prec}(t), \partial_1 C_{\beta,e}(t, y)\big]$. Applying (3.3) to α and β yields

$$\begin{aligned} \int_0^1 V_{A_t}(S(t))\,\mathrm{d}t &= \int_0^{\alpha(x)} V_{A_t}\big([\mu_p^{\prec}(t), \mu_p^{\preceq}(t)] \times [\eta_q^{\prec}(t), \eta_q^{\preceq}(t)]\big)\,\mathrm{d}t \\ &\quad + \int_{\alpha(x)}^1 V_{A_t}\big([\mu_p^{\prec}(t), \mu_p^{\prec}(t)] \times [\eta_q^{\prec}(t), \eta_q^{\prec}(t)]\big)\,\mathrm{d}t \\ &= \int_0^{\alpha(x)} V_{A_t}(R_{p,q}(t))\,\mathrm{d}t. \end{aligned}$$

The other three cases where α_p or β_q is decreasing can be proved similarly. ☐

Lemma 9. *With the same notations as in Theorem 7, if* $(B_t)_{t\in\mathbb{I}}$ *and* $(C_t)_{t\in\mathbb{I}}$ *are such that* $C_{e,\alpha} *_{(A_t)} C_{\beta,e} = \frac{1}{2} C_{e,\alpha} *_{(B_t)} C_{\beta,e} + \frac{1}{2} C_{e,\alpha} *_{(C_t)} C_{\beta,e}$*, then* $T(A_t) = \frac{1}{2} T(B_t) + \frac{1}{2} T(C_t)$ *a.e.*

Proof. By the correspondence between copulas and doubly stochastic measures, $\mu_{D(A)}(S) = \frac{1}{2}\mu_{D(B)}(S) + \frac{1}{2}\mu_{D(C)}(S)$ where $D(E) := C_{e,\alpha} *_{(E_t)} C_{\beta,e}$, for $E \in \{A, B, C\}$, and $S := [a, b] \times [c, d] \subseteq \mathbb{I}^2$. It then follows from Proposition 8 that for $(p, q) \in \mathcal{I} \times \mathcal{J}$ and $x \in I_p$,

$$\int_0^{\alpha(x)} V_{A_t}(R_{p,q}(t))\,\mathrm{d}t = \frac{1}{2}\int_0^{\alpha(x)} V_{B_t}(R_{p,q}(t))\,\mathrm{d}t + \frac{1}{2}\int_0^{\alpha(x)} V_{C_t}(R_{p,q}(t))\,\mathrm{d}t.$$

Since $\alpha|_{I_p}$ is surjective, $V_{A_t}(R_{p,q}(t)) = \frac{1}{2} V_{B_t}(R_{p,q}(t)) + \frac{1}{2} V_{C_t}(R_{p,q}(t))$ for a.e. t. Hence, $T(A_t) = \frac{1}{2} T(B_t) + \frac{1}{2} T(C_t)$ a.e. as desired. ☐

We are ready to prove Theorem 7.

Proof of Theorem 7. ($\Rightarrow$) Suppose that, for t in a Borel set E of positive measure, there exist distinct $[b_{p,q}(t)]$ and $[c_{p,q}(t)]$ in $\mathcal{M}\left(P(t), Q(t)\right)$ whose average is equal to $T(A_t)$. Let us define copulas B_t and C_t for each $t \in \mathbb{I}$ as follows. If $t \in E$, let B_t and C_t be the copulas satisfying (3.5) for the transformation matrices $[b_{p,q}(t)]$ and $[c_{p,q}(t)]$, respectively. Their existence is guaranteed by the bilinear interpolation. If $t \notin E$, let $B_t = C_t := A_t$. From the construction, $T(B_t) = [b_{p,q}(t)]$ and $T(C_t) = [c_{p,q}(t)]$ for $t \in E$ and $T(A_t) = \frac{1}{2}T(B_t) + \frac{1}{2}T(C_t)$ for a.e. t. So $\mu_C = \frac{1}{2}\mu_{C_{e,\alpha} *_{(B_t)} C_{\beta,e}} + \frac{1}{2}\mu_{C_{e,\alpha} *_{(C_t)} C_{\beta,e}}$ and it remains to show that $\mu_{C_1} := \mu_{C_{e,\alpha} *_{(B_t)} C_{\beta,e}}$ and $\mu_{C_2} := \mu_{C_{e,\alpha} *_{(C_t)} C_{\beta,e}}$ are not equal. Since $B_t \neq C_t$ for all $t \in E$ and $\mathcal{I} \times \mathcal{J}$ is countable, there exists an index $(p, q) \in \mathcal{I} \times \mathcal{J}$ and a measurable subset $E' \subseteq E$ of positive measure such that $a_{p,q}(t) = \frac{1}{2}b_{p,q}(t) + \frac{1}{2}c_{p,q}(t)$ where $b_{p,q}(t) \neq c_{p,q}(t)$ for $t \in E'$. This implies that there is a rectangle $S \subseteq I_p \times J_q$, defined as in the proof of Proposition 8, satisfying $\mu_{C_1}(S) - \mu_{C_2}(S) = \int_{[0,\alpha(x)]\cap E'} \left(b_{p,q}(t) - c_{p,q}(t)\right) \mathrm{d}\lambda(t) \neq 0$. So $\mu_{C_1} \neq \mu_{C_2}$ and $C = C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ is not extreme.

($\Leftarrow$) Suppose that $\mu_C = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ for some doubly stochastic measures μ_1, μ_2 with $\mu_1 \neq \mu_2$. Then $\mu_i \ll \mu_C$ for $i = 1, 2$ and, by Lemma 3, $\mu_1 = \mu_{C_{e,\alpha} *_{(B_t)} C_{\beta,e}}$ and $\mu_2 = \mu_{C_{e,\alpha} *_{(C_t)} C_{\beta,e}}$ for some $(B_t)_{t\in\mathbb{I}}, (C_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}$. By Lemma 9, $T(A_t) = \frac{1}{2}T(B_t) + \frac{1}{2}T(C_t)$ a.e. t. Since $\mu_1 \neq \mu_2$, there is an index $(p, q) \in \mathcal{I} \times \mathcal{J}$ and a rectangle $S \subseteq I_p \times J_q$ chosen as in the proof of Proposition 8 such that $\mu_1(S) \neq \mu_2(S)$, i.e., $\int_0^{\alpha(x)} b_{p,q}(t)\,\mathrm{d}t = \int_0^1 V_{B_t}\left(S(t)\right)\mathrm{d}t \neq \int_0^1 V_{C_t}\left(S(t)\right)\mathrm{d}t = \int_0^{\alpha(x)} c_{p,q}(t)\,\mathrm{d}t$ for some $x \in I_p$. Consequently, there is a measurable set $E \subseteq \mathbb{I}$ with $\lambda(E) > 0$ such that $b_{p,q}(t) \neq c_{p,q}(t)$, i.e., $T(B_t) \neq T(C_t)$, for every $t \in E$. Therefore, $T(A_t)$ is not extreme on E. $\square$

Example 10. Let $\alpha, \beta \in \mathscr{T}_{\text{CPMS}}$ be defined, as shown in Figure 5, by

$$\alpha(x) = \begin{cases} 3x & \text{if } 0 \leqslant x \leqslant \frac{1}{6}; \\ 2x + \frac{1}{6} & \text{if } \frac{1}{6} < x \leqslant \frac{5}{12}; \\ \frac{11}{6} - 2x & \text{if } \frac{5}{12} < x \leqslant \frac{2}{3}; \\ \frac{3}{2} - \frac{3}{2}x & \text{if } \frac{2}{3} < x \leqslant 1 \end{cases} \quad \text{and} \quad \beta(y) = \begin{cases} 1 - 4y & \text{if } 0 \leqslant y \leqslant \frac{1}{4}; \\ 2y - \frac{1}{2} & \text{if } \frac{1}{4} < y \leqslant \frac{3}{4}; \\ 4y - 3 & \text{if } \frac{3}{4} < y \leqslant 1. \end{cases}$$

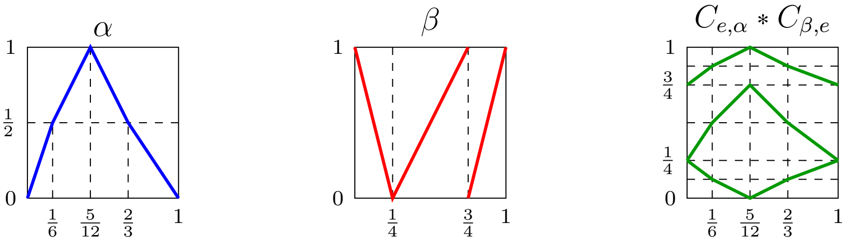

Figure 5: Graphs of functions α and β and the support of $C_{e,\alpha} * C_{\beta,e}$ in Example 10

Note that $P(t) = \begin{cases} \left\{\left(0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right)\right\} & \text{if } 0 < t < \frac{1}{2}; \\ \left\{\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right)\right\} & \text{if } \frac{1}{2} < t < 1, \end{cases}$ and $Q(t) = \left\{\left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, 1\right)\right\}$ for all $t \in \mathbb{I}$, which generate a collection of 2×3 transformation matrices in $\mathcal{M}(P(t), Q(t))$ for a given family of joining copulas $(A_t)_{t\in\mathbb{I}}$.

- If $A_t \equiv \Pi$ then $T(A_t)$ is not extreme in $\mathcal{M}(P(t), Q(t))$ for a.e. t. In fact,

$$T(A_t) = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix} \text{ for } t \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad T(A_t) = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix} \text{ for } t \in \left(\frac{1}{2}, 1\right).$$

- If $A_t \equiv C_{e,s_1}$ where $s_1(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \leqslant x \leqslant \frac{1}{2}; \\ x - \frac{1}{2} & \text{if } \frac{1}{2} < x \leqslant 1, \end{cases}$ then, in $\mathcal{M}(P(t), Q(t))$, $T(A_t)$ is not extreme for $t \in \left(0, \frac{1}{2}\right)$ but extreme for $t \in \left(\frac{1}{2}, 1\right)$. In fact

$$T(A_t) = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix} \text{ for } t \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad T(A_t) = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix} \text{ for } t \in \left(\frac{1}{2}, 1\right).$$

- If $A_t \equiv C_{e,s_2}$ where $s_2(x) = \begin{cases} 1 - x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}; \\ x - \frac{1}{2} & \text{if } \frac{1}{2} < x \leqslant 1, \end{cases}$ then $T(A_t)$ is extreme in $\mathcal{M}(P(t), Q(t))$ for a.e. t. In fact,

$$T(A_t) = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{12} & \frac{5}{12} \\ 0 & \frac{1}{4} \end{bmatrix} \text{ for } t \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad T(A_t) = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix} \text{ for } t \in \left(\frac{1}{2}, 1\right).$$

The graphs of s_1, s_2 and the supports of $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ where $A_t = C_{e,s_1}$ and C_{e,s_2} for a.e. $t \in \mathbb{I}$ are shown in Figure 6. In the supports of $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$, the orange and green segments represent extreme and non-extreme parts, respectively.

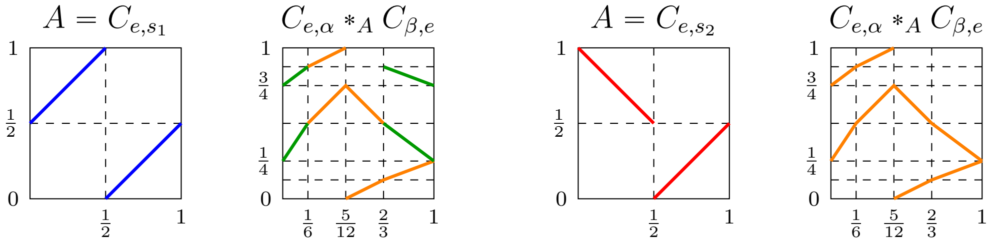

Figure 6: Supports of joining copula A and the corresponding product $C_{e,\alpha} *_A C_{\beta,e}$ for α and β discussed in Example 10 and shown in Figure 5, where (left) $A = C_{e,s_1}$ and (right) $A = C_{e,s_2}$

4 Characterization of extreme implicit dependence copulas

Outcomes in Section 3 are combined to build a set of sufficient conditions (Theorem 15) for the extremality of copulas in $\mathscr{C}_{\mathrm{ID}}$. This then hints on how to set up a general condition (Theorem 19) that characterizes extreme implicit dependence copulas. Let $C \in \mathscr{C}_{\mathrm{ID}}$ be such that $\alpha(U) = \beta(V)$ a.s. for some $\alpha, \beta \in \mathscr{T}$ and $U, V \sim \mathcal{U}(0,1)$. By Lemma 1, μ_C can be split into several parts according to the values of $\alpha(U)$ (or $\beta(V)$).

Definition 11. Let $\alpha \in \mathscr{T}$. Define the following parts in the range of α.

- The *injective part* of α is defined as $\mathbb{I}_\alpha := \{t \in \mathbb{I}\colon \operatorname{card}(\alpha^{-1}(t)) = 1\}$, where $\operatorname{card}(A)$ is the cardinality of A. Without loss of generality, $\mathbb{I}_\alpha$ is assumed to be a Borel set.

- An open interval B is called a *CPMS interval* of α if $\alpha\,|_{\alpha^{-1}(B)}\colon \alpha^{-1}(B) \to B$ is a CPMS function in $\mathscr{T}\left(\alpha^{-1}(B), B\right)$. That is, there exists a countable set $\mathcal{I}$ with at least two elements and a partition (a.e.) $P_B := \{I_n\}_{n\in\mathcal{I}}$ of $\alpha^{-1}(B)$ consisting of open intervals such that $\alpha|_{I_n}$ is a strictly monotonic function from I_n onto B. And a *CPMS part* is a maximal CPMS interval whose existence is guaranteed by Zorn's lemma. It is evident that the Borel set $\mathbb{I}_\alpha^{\mathrm{CPMS}} := \dot{\bigcup}\{B\colon B \text{ is a CPMS part of } \alpha\}$ and $\mathbb{I}_\alpha$ are disjoint.

Clearly, an open subinterval of a CPMS interval is also a CPMS interval. Moreover, there are at most countably many CPMS parts.

Recall [32] that if $t \in \mathbb{I}_\alpha$ then $\partial_2 C_{e,\alpha}(x,t) = \mathbb{1}_{[0,x]}\left(\alpha^{-1}(t)\right)$ a.e. In fact, the converse also holds.

Lemma 12. *Let $\psi \in \mathscr{T}$ and $A \in \mathscr{B}$ with $\lambda(A) > 0$. Then $T_\psi^* \mathbb{1}_A = 0$ a.e. on $\mathbb{I}\backslash\psi(A)$ and $T_\psi^*\mathbb{1}_A > 0$ a.e. on $\psi(A)$.*

Proof. By the measurability of $\psi(A)$, there exists $D \in \mathscr{B}$ containing $\psi(A)$ with $\lambda(D) = \lambda\left(\psi(A)\right)$. So if $B \in \mathscr{B}$ is such that $\lambda\left(B \cap \psi(A)\right) = 0$, then $\lambda\left(B \cap \psi(A)\right) = \lambda\left(B \cap D\right) = \lambda\left(\psi^{-1}(B) \cap \psi^{-1}(D)\right) \geqslant \lambda\left(\psi^{-1}(B) \cap A\right)$, i.e., $\lambda\left(A \cap \psi^{-1}(B)\right) = 0$. Hence

$$\int_B T_\psi^* \mathbb{1}_A \,\mathrm{d}\lambda = \int_{\mathbb{I}} \mathbb{1}_A \cdot T_\psi \mathbb{1}_B \,\mathrm{d}\lambda = \int_{\mathbb{I}} \mathbb{1}_{A\cap\psi^{-1}(B)} \,\mathrm{d}\lambda = 0.$$

Since B is an arbitrary Borel subset of $\mathbb{I}\backslash\psi(A)$, $T_\psi^*\mathbb{1}_A = 0$ a.e. on $\mathbb{I}\backslash\psi(A)$.

To show the second part, suppose that B is a positive-measure Borel subset of $\psi(A)$ for which $\lambda\left(A \cap \psi^{-1}(B)\right) = 0$. This implies that $A \subseteq \psi^{-1}(B)\dot{\cup}\psi^{-1}\left(\psi(A)\backslash B\right) \subseteq \psi^{-1}\left(\psi(A)\backslash B\right)$ a.e. and then $B \subseteq \psi(A) \subseteq \psi(A)\backslash B$ a.e. which is a contradiction. Hence, $\int_B T_\psi^*\mathbb{1}_A \,\mathrm{d}\lambda = \lambda\left(A \cap \psi^{-1}(B)\right) > 0$ for all $B \subseteq \psi(A)$. Thus, $T_\psi^*\mathbb{1}_A > 0$ a.e. on $\psi(A)$. ☐

Lemma 13. *For $\alpha \in \mathscr{T}$, it holds for a.e. t that $t \in \mathbb{I}_\alpha$ if and only if $\partial_2 C_{e,\alpha}(x,t) \in \{0,1\}$ for a.e. $x \in \mathbb{I}$.*

Proof. The "only if" part is shown in [32, Lemma 7]. For the "if" part, it suffices to show that $A_\alpha \subseteq \mathbb{I}_\alpha$ a.e., i.e., $\lambda\left(A_\alpha \cap \mathbb{I}_\alpha\right) = \lambda\left(A_\alpha\right)$, where $A_\alpha := \{t \in \mathbb{I}\colon \partial_2 C_{e,\alpha}(x,t) \in \{0,1\} \text{ a.e. } x\}$. For each $t \in A_\alpha$, since $\partial_2 C_{e,\alpha}(x,t)$ is increasing in x, there exists $g(t) \in \mathbb{I}$ such that

$$T_\alpha^*\mathbb{1}_{[0,x]}(t) = \partial_2 C_{e,\alpha}(x,t) = \mathbb{1}_{[g(t),1]}(x) = \mathbb{1}_{[0,x]} \circ g(t) \qquad \text{a.e. } x. \tag{4.1}$$

We then claim that, for a.e. x and a.e. $a \in \alpha^{-1}\left(A_\alpha\right)$, $\mathbb{1}_{[0,x]}\left(g(\alpha(a))\right) = \mathbb{1}_{[0,x]}(a)$, which yields that $t \in \mathbb{I}_\alpha$. In fact, if $t = \alpha(a) = \alpha(b)$ then $\mathbb{1}_{[0,x]}(a) = \partial_2 C_{e,\alpha}(x,t) = \mathbb{1}_{[0,x]}(b)$ a.e. x and $a = b$. To prove the claim, suppose $a \in [0,x]$. Then $\alpha(a) \in \alpha[0,x]$ and, by Lemma 12, $T_\alpha^*\mathbb{1}_{[0,x]}\left(\alpha(a)\right) > 0$. By (4.1), $\mathbb{1}_{[0,x]} \circ g\left(\alpha(a)\right) = 1$, i.e., $g(\alpha(a)) \in [0,x]$. Conversely, if $g(\alpha(a)) \in [0,x]$, then $1 = \mathbb{1}_{[0,x]}\circ g\left(\alpha(a)\right) = T_\alpha^*\mathbb{1}_{[0,x]}\left(\alpha(a)\right) = 1 - T_\alpha^*\mathbb{1}_{(x,1]}\left(\alpha(a)\right)$. Therefore, $T_\alpha^*\mathbb{1}_{(x,1]}\left(\alpha(a)\right) = 0$ and, by Lemma 12, $\alpha(a) \notin \alpha(x,1]$. This implies that $a \notin (x,1]$, that is $a \in [0,x]$. ☐

On the other hand, if $t \in \mathbb{I}_\alpha^{\mathrm{CPMS}}$, then the value of $\partial_2 C_{e,\alpha}(x,t)$ satisfies the following statement.

Lemma 14. *Let an open interval B be a CPMS part of $\alpha \in \mathscr{T}$, i.e., there exists a partition (a.e.) $P_B := \{I_n := (a_n,b_n)\}_{n\in\mathcal{I}}$ of $\alpha^{-1}(B)$ such that each $\alpha|_{I_n}$ is a strictly monotonic function from I_n onto B. Then for $p \in \mathcal{I}$ and a.e. $(x,t) \in I_p \times B$,*

$$\partial_2 C_{e,\alpha}(x,t) = \begin{cases} \mu_p^{<}(t) & \text{if } x < \alpha_p^{-1}(t); \\ \mu_p^{\leq}(t) & \text{if } x > \alpha_p^{-1}(t), \end{cases}$$

where $\mu_p^<$ *and* $\mu_p^\leq$ *are defined in Subsection 3.2. Otherwise, for a.e.* $(x,t) \in (\mathbb{I} \setminus \alpha^{-1}(B)) \times B$, $\partial_2 C_{e,\alpha}(x,t) = \mu_p^\leq(t) = \mu_q^<(t)$ *for a unique pair* $p, q \in \mathcal{I}$ *satisfying* $b_p < x < a_q$. *Let* $b_p = 0$ *if* I_q *is the leftmost interval, and* $a_q = 1$ *if* I_p *is the rightmost interval in* $\alpha^{-1}(B)$.

Proof. It follows directly from (2.1) that $T_\alpha^* \mathbb{1}_{[0,x]}(t) = \partial_1 C_{\alpha,e}(t,x) = \partial_2 C_{e,\alpha}(x,t)$, which, by [14], is equal to $\sum_{i\in\mathcal{I}} \frac{1}{|\alpha'(\alpha_i^{-1}(t))|} \mathbb{1}_{[0,x]}\left(\alpha_i^{-1}(t)\right)$ for a.e. $t \in B$ and $x \in \mathbb{I}$. Hence the result follows from the definition of $\mu_n^<(t)$ and $\mu_n^\leq(t)$. □

By combining the main result in [32], Theorem 4 and Theorem 7, we obtain broader sufficient conditions for implicit dependence copulas to be extreme.

Theorem 15. *Let* $\alpha, \beta \in \mathscr{T}$ *and* $(A_t)_{t\in\mathbb{I}}$ *be a collection of joining copulas satisfying the following properties.*

1. *For a.e.* $t \in \mathbb{I}_0 := \mathbb{I}_\alpha \cup \mathbb{I}_\beta$, $A_t\left(\partial_2 C_{e,\alpha}(x,t), \partial_1 C_{\beta,e}(t,y)\right)$ *is measurable in* t *for a.e.* $(x,y) \in \mathbb{I}^2$.
2. *For a.e.* $t \in \mathbb{I}_1 := \mathbb{I}_\alpha^{CPMS} \cap \mathbb{I}_\beta^{CPMS}$, $T(A_t) := \left[V_{A_t}\left(I_p'(t) \times J_q'(t)\right)\right]_{(p,q)\in\mathcal{I}\times\mathcal{J}}$ *is extreme in* $\mathcal{M}\left(P(t), Q(t)\right)$. *Here,* $P(t) = \left\{I_p'(t)\right\}_{p\in\mathcal{I}}$ *and* $Q(t) = \left\{J_q'(t)\right\}_{q\in\mathcal{J}}$ *as in (3.4) are defined from* $\alpha|_{\alpha^{-1}(B)}$ *and* $\beta|_{\beta^{-1}(B)}$, *respectively, where* B *is a CPMS part in* $\mathbb{I}_1$.
3. *There is a Borel set* $S \subseteq \mathbb{I} \setminus (\mathbb{I}_0 \cup \mathbb{I}_1)$ *such that* $A_t = W$ *for a.e.* $t \in S$ *and* $A_t = M$ *for* $t \in \mathbb{I} \setminus (\mathbb{I}_0 \cup \mathbb{I}_1 \cup S)$.

Then $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ *is extreme.*

Proof. By Lemma 1, $\mu_C = \mu_{C,\mathbb{I}_0} + \mu_{C,\mathbb{I}_1} + \mu_{C,\mathbb{I}\setminus(\mathbb{I}_0\cup\mathbb{I}_1)}$ where $\mu_{C,E}$ is defined by the integral in (3.1). We then show that the three components are extreme.

- $\mu_{C,\mathbb{I}_0}$: Since $\partial_2 C_{e,\alpha}(x,t), \partial_1 C_{\beta,e}(s,x) \in \{0,1\}$ for a.e. $t \in \mathbb{I}_\alpha$, a.e. $s \in \mathbb{I}_\beta$ and $x \in \mathbb{I}$,
$$V_{A_t}\left(R(t)\right) = \left(\partial_2 C_{e,\alpha}(b,t) - \partial_2 C_{e,\alpha}(a,t)\right)\left(\partial_1 C_{\beta,e}(t,d) - \partial_1 C_{\beta,e}(t,c)\right) = V_\Pi\left(R(t)\right)$$
for a.e. $t \in \mathbb{I}_0$ where $R := [a,b] \times [c,d] \subseteq \mathbb{I}^2$ and $R(t) := [\partial_2 C_{e,\alpha}(a,t), \partial_2 C_{e,\alpha}(b,t)] \times [\partial_1 C_{\beta,e}(t,c), \partial_1 C_{\beta,e}(t,d)]$. This gives $\mu_{C,\mathbb{I}_0}(R) = \mu_{C_{e,\alpha}*C_{\beta,e},\mathbb{I}_0}(R)$ and so, by a similar proof to that in [32], $\mu_{C,\mathbb{I}_0}$ is extreme.
- $\mu_{C,\mathbb{I}_1}$: Clearly, $\mathbb{I}_1$ is the union of (non-empty) intersections of CPMS parts of α and β. So $\mu_{C,\mathbb{I}_1}$ is the sum of $\mu_{C,B_i^\alpha\cap B_j^\beta}$ where B_i^α and B_j^β are CPMS parts of α and β, respectively. By a variant of Theorem 7 for measures on $\alpha^{-1}(B) \times \beta^{-1}(B)$, each $\mu_{C,B}$ is extreme for $B = B_i^\alpha \cap B_j^\beta$.
- $\mu_{C,\mathbb{I}\setminus(\mathbb{I}_0\cup\mathbb{I}_1)}$: Applying a variant of Theorem 4 for measures on $\alpha^{-1}\left(\mathbb{I}\setminus(\mathbb{I}_0\cup\mathbb{I}_1)\right) \times \beta^{-1}\left(\mathbb{I}\setminus(\mathbb{I}_0\cup\mathbb{I}_1)\right)$, the assumption yields the extremality of $\mu_{C,\mathbb{I}\setminus(\mathbb{I}_0\cup\mathbb{I}_1)}$. □

Furthermore, if $\alpha, \beta \in \mathscr{T}$ satisfy $\lambda\left(\mathbb{I}_\alpha \cup \mathbb{I}_\alpha^{\text{CPMS}}\right) = \lambda\left(\mathbb{I}_\beta \cup \mathbb{I}_\beta^{\text{CPMS}}\right) = 1$, then $\lambda\left(\mathbb{I}_0 \cup \mathbb{I}_1\right) = 1$ and the extremality of $T(A_t)$ is a necessary condition for the extremality of $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$.

Corollary 16. *Let* $\alpha, \beta \in \mathscr{T}$ *be such that* $\lambda\left(\mathbb{I}_\alpha \cup \mathbb{I}_\alpha^{CPMS}\right) = \lambda\left(\mathbb{I}_\beta \cup \mathbb{I}_\beta^{CPMS}\right) = 1$ *and* $(A_t)_{t\in\mathbb{I}}$ *be a collection of joining copulas. Then* $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ *is extreme if and only if* $T(A_t)$ *is extreme in* $\mathcal{M}\left(P(t), Q(t)\right)$ *for a.e.* $t \in \mathbb{I} \setminus (\mathbb{I}_\alpha \cup \mathbb{I}_\beta)$.

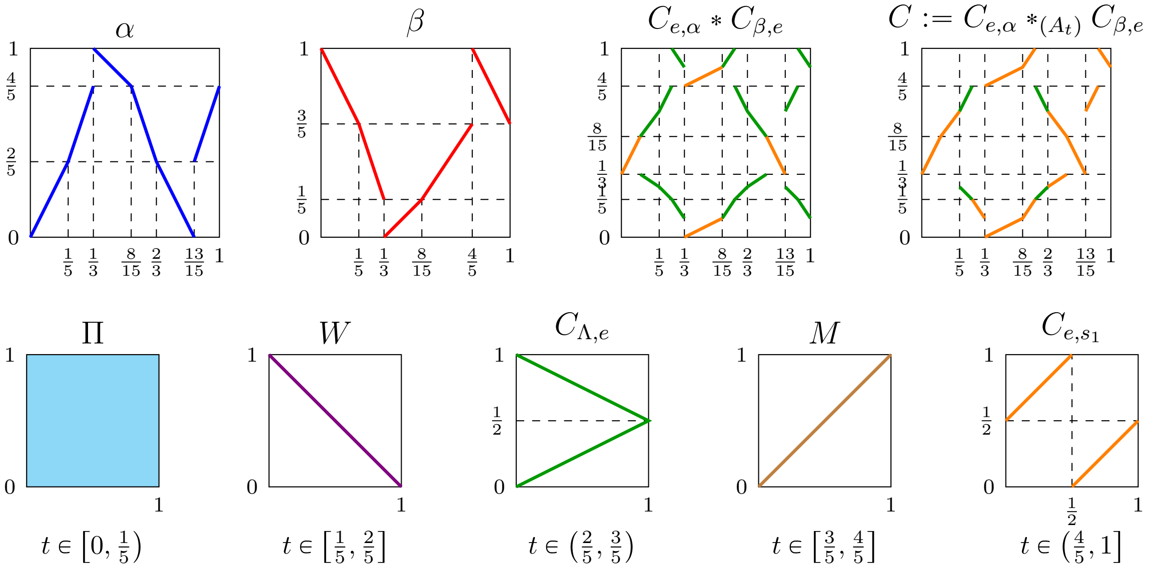

Figure 7: (top) Graphs of α, β as well as supports of $C_{e,\alpha} * C_{\beta,e}$ and C; (bottom) supports of joining copulas A_t referred in Example 17

Example 17. Let α, β be piecewise linear functions in $\mathscr{T}$ and $(A_t)_{t\in\mathbb{I}}$ a collection of joining copulas as shown in Figure 7. Observe that $\mathbb{I}_1 = \mathbb{I}_\alpha^{\text{CPMS}} \cap \mathbb{I}_\beta^{\text{CPMS}} = \left(\frac{1}{5}, \frac{4}{5}\right)$ and $\mathbb{I}_0 = \mathbb{I}_\alpha \cup \mathbb{I}_\beta = \left[0, \frac{1}{5}\right) \cup \left(\frac{4}{5}, 1\right]$ and $A_t = W\mathbb{1}_{\left[\frac{1}{5},\frac{2}{5}\right]}(t) + M\mathbb{1}_{\left[\frac{3}{5},\frac{4}{5}\right]}(t)$. According to Theorem 15, the extremality of μ_C pivots on that of $\mu_{C,\left(\frac{2}{5},\frac{3}{5}\right)}$. For $t \in \left(\frac{2}{5}, \frac{3}{5}\right)$,

$$P(t) = \left\{\left(0, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, 1\right)\right\}, \; Q(t) = \left\{\left(0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right)\right\} \; \text{ and } \; T(A_t) = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix},$$

which is not extreme in $\mathcal{M}(P(t), Q(t))$. Hence, $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ is not extreme. The support of $C_{e,\alpha} * C_{\beta,e}$ and C are shown in Figure 7 in which the orange and green segments represent extreme and non-extreme parts, respectively. Obviously, $T(A_t)$ are all extreme in $\mathcal{M}(P(t), Q(t))$ for all $t \in \mathbb{I} \setminus \left(\frac{2}{5}, \frac{3}{5}\right)$.

To generalize Theorem 15, let us denote for each $C \in \mathscr{C}$

$$D_C(t) := \{\partial_2 C(x, t) : x \in \mathbb{I}\} \qquad \text{for each } t \in \mathbb{I}, \tag{4.2}$$

and $D_\alpha := D_{C_{e,\alpha}}$ for $\alpha \in \mathscr{T}$. Note that $\{\partial_1 C_{\beta,e}(t, y) : y \in \mathbb{I}\} = D_\beta$ as well. Since every transformation matrix with respect to a partition induces a subcopula on its boundary and vice versa as stated in the paragraph before Theorem 7, in the general case where $D_\alpha(t)$ or $D_\beta(t)$ might be uncountable, it is natural to consider instead the extremality of A_t as a subcopula on $D_\alpha(t) \times D_\beta(t)$. Here, we denote by $\mathcal{S}(A \times B)$ the class of subcopulas on $A \times B$ where $\{0, 1\} \subseteq A, B \subseteq \mathbb{I}$.

Remark 18. From Theorem 15, we observe that for $\alpha, \beta \in \mathscr{T}$,

1. if $t \in \mathbb{I}_0$, then $D_\alpha(t)$ or $D_\beta(t)$ becomes $\{0, 1\}$ by Lemma 13 which trivially gives the extremality of A_t in $\mathcal{S}(D_\alpha(t) \times D_\beta(t))$;
2. if $t \in \mathbb{I}_1$, then we can find A_t on $D_\alpha(t) \times D_\beta(t) = \partial P(t) \times \partial Q(t)$ via (3.5). Hence, $T(A_t)$ is extreme in $\mathcal{M}(P(t), Q(t))$ if and only if A_t is extreme in $\mathcal{S}(D_\alpha(t) \times D_\beta(t))$; and

3. if $t \in \mathbb{I} \setminus (\mathbb{I}_0 \cup \mathbb{I}_1)$, then A_t is extreme in $\mathcal{S}(D_\alpha(t) \times D_\beta(t))$ because A_t is a Fréchet-Hoeffding bound for a.e. $t \in \mathbb{I} \setminus (\mathbb{I}_0 \cup \mathbb{I}_1)$.

Thus, every assumption in Theorem 15 can be fused into one statement that "A_t is extreme in $\mathcal{S}(D_\alpha(t) \times D_\beta(t))$." In fact, this is a characterization of the extremality of $C_{e,\alpha} *_{(A_t)} C_{\beta,e}$. Its proof is adapted from the proof of Theorem 7.

Theorem 19 (Characterization of extreme bivariate IDCs)**.** *Let $(A_t)_{t\in\mathbb{I}}$ be a collection of joining copulas and $E, F \in \mathscr{C}$. If $E *_{(A_t)} F$ is extreme then A_t is extreme in $\mathcal{S}(D_E(t) \times D_{F^T}(t))$ for a.e. $t \in \mathbb{I}$. Moreover, if $E = C_{e,\alpha}$ and $F = C_{\beta,e}$ for some $\alpha, \beta \in \mathscr{T}$ then the extremality of A_t a.e. is a necessary and sufficient condition, i.e., $C := C_{e,\alpha} *_{(A_t)} C_{\beta,e}$ is extreme if and only if A_t is extreme in $\mathcal{S}(D_\alpha(t) \times D_\beta(t))$ for a.e. $t \in \mathbb{I}$.*

Proof. ($\Rightarrow$) Assume that there exists $P \subseteq \mathbb{I}$ of positive measure for which $A_t = \frac{1}{2}B_t + \frac{1}{2}C_t$ for some $B_t, C_t \in \mathscr{C}$ with $B_t \neq C_t$ on $D_\alpha(t) \times D_\beta(t)$ for all $t \in P$. Without loss of generality, there exists $(a,b) \in \mathbb{I}^2$ for which the set $P_{(a,b)} := \left\{t \in P : B_t\left(R_{(a,b)}(t)\right) > C_t\left(R_{(a,b)}(t)\right)\right\}$ has positive Lebesgue measure, where $R_{(a,b)}(t) := (\partial_2 C_{e,\alpha}(a,t), \partial_1 C_{\beta,e}(t,b))$. If we define two new classes of joining copulas by

$$B'_t := \begin{cases} B_t & \text{if } t \in P_{(a,b)}; \\ A_t & \text{otherwise,} \end{cases} \quad \text{and} \quad C'_t := \begin{cases} C_t & \text{if } t \in P_{(a,b)}; \\ A_t & \text{otherwise,} \end{cases}$$

then $C_1 := C_{e,\alpha} *_{(B'_t)} C_{\beta,e}$ and $C_2 := C_{e,\alpha} *_{(C'_t)} C_{\beta,e}$ are not equal as

$$C_1(a,b) - C_2(a,b) = \int_{P_{(a,b)}} \left[B'_t\left(R_{(a,b)}(t)\right) - C'_t\left(R_{(a,b)}(t)\right)\right] \mathrm{d}\lambda(t) > 0.$$

Moreover, the assumption implies that $A_t = \frac{1}{2}B'_t + \frac{1}{2}C'_t$ on $D_\alpha(t) \times D_\beta(t)$ for all $t \in \mathbb{I}$ and hence $C = \frac{1}{2}C_1 + \frac{1}{2}C_2$.

($\Leftarrow$) Assume that $\mu_C = \frac{1}{2}\mu_{C_1} + \frac{1}{2}\mu_{C_2}$ for some $C_1, C_2 \in \mathscr{C}$ with $C_1 \neq C_2$. Then $\mu_{C_i} \ll \mu_C$ for $i = 1, 2$ and hence, by Lemma 3, $C_1 = C_{e,\alpha} *_{(B_t)} C_{\beta,e}$ and $C_2 = C_{e,\alpha} *_{(C_t)} C_{\beta,e}$ for some $(B_t)_{t\in\mathbb{I}}, (C_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}$. Restricting the above measures on $\mathrm{Gr}_{\alpha,\beta}(S)$ for $S \in \mathscr{B}$ (see (3.2)), we obtain $\mu_{C,S} = \frac{1}{2}\mu_{C_1,S} + \frac{1}{2}\mu_{C_2,S}$. In particular, for $(x,y) \in \mathbb{I}^2$,

$$\begin{aligned} \int_S A_t\left(R_{(x,y)}(t)\right) \mathrm{d}\lambda(t) &= \frac{1}{2}\int_S B_t\left(R_{(x,y)}(t)\right) \mathrm{d}\lambda(t) + \frac{1}{2}\int_S C_t\left(R_{(x,y)}(t)\right) \mathrm{d}\lambda(t) \\ &= \int_S \left(\frac{1}{2}B_t + \frac{1}{2}C_t\right)\left(R_{(x,y)}(t)\right) \mathrm{d}\lambda(t). \end{aligned}$$

Since S and (x,y) are arbitrary, we have $A_t = \frac{1}{2}B_t + \frac{1}{2}C_t$ on $D_\alpha(t) \times D_\beta(t)$ for a.e. $t \in \mathbb{I}$. Finally, the fact that $B_t \neq C_t$ on $D_\alpha(t) \times D_\beta(t)$ for all t in a set of positive measure P can be verified in the similar way as Theorem 7. ☐

5 Extreme multivariate implicit dependence copulas

In this context, (d-dimensional) implicit dependence copulas are joint distribution functions of $\mathcal{U}(0,1)$-random variables $X_1, X_2, \ldots, X_d$ for which $\alpha_1(X_1) = \alpha_2(X_2) = \cdots = \alpha_d(X_d)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathscr{T}$. From [1, 31], they can be written as a d-dimensional generalized product of C_{e,α_i}, i.e., $\mathop{\ast_{\mathcal{A}}}\limits_{i=1}^{d} C_{e,\alpha_i}\colon (x_1,\dots,x_d)\mapsto \int_0^1 A_t\left(\left(\partial_2 C_{e,\alpha_i}(x_i,t)\right)_{i=1}^d\right)\, dt$ for some $\mathcal{A} := (A_t)_{t\in\mathbb{I}} \subseteq \mathscr{C}_d$, the space of d-copulas. As in 2-dimensional case, $\ast_{i=1}^d C_{e,\alpha_i} := \mathop{\ast_{(\Pi_d)}}\limits_{i=1}^{d} C_{e,\alpha_i}$, where $\Pi_d\colon (x_1,\dots,x_d)\mapsto x_1\cdots x_d$ is the d-dimensional independent copula, are called *factorizable copulas*. The proof in [32] can be adapted to multivariate case and gives a necessary and sufficient condition for factorizable d-copulas to be extreme.

Theorem 20. *For $\alpha_1,\alpha_2,\dots,\alpha_d\in\mathscr{T}$, $\ast_{i=1}^d C_{e,\alpha_i}$ is extreme if and only if $\lambda\left(\mathbb{I}_{\alpha_i}\cup\mathbb{I}_{\alpha_j}\right)=1$ for all $i\neq j$.*

Example 21. Let $\alpha,\beta,\gamma_1,\gamma_2\in\mathscr{T}$ whose graphs are shown in Figure 8. Then the support of factorizable copulas $C_1 := \ast\left(C_{e,\alpha},C_{e,\beta},C_{e,\gamma_1}\right)$ and $C_2 := \ast\left(C_{e,\alpha},C_{e,\beta},C_{e,\gamma_2}\right)$ lie on the graphs $\alpha(x)=\beta(y)=\gamma_1(z)$ and $\alpha(x)=\beta(y)=\gamma_2(z)$, respectively, as shown in Figure 9. Observe that $\lambda\left(\mathbb{I}_\alpha\cup\mathbb{I}_\beta\right)=\lambda\left(\mathbb{I}_\alpha\cup\mathbb{I}_{\gamma_1}\right)=\lambda\left(\mathbb{I}_\beta\cup\mathbb{I}_{\gamma_1}\right)=1$ but $\lambda\left(\mathbb{I}_\alpha\cup\mathbb{I}_{\gamma_2}\right)=\lambda\left(\mathbb{I}_\beta\cup\mathbb{I}_{\gamma_2}\right)=\frac{3}{4}<1$. Hence by Theorem 20, C_1 is extreme while C_2 is not. In addition, it is straightforward to show that

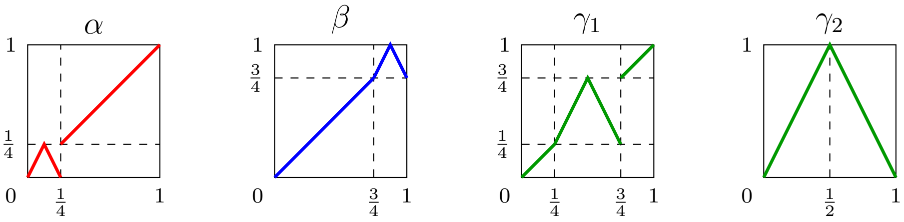

Figure 8: Graphs of α,β,γ_1 and γ_2 in Example 21

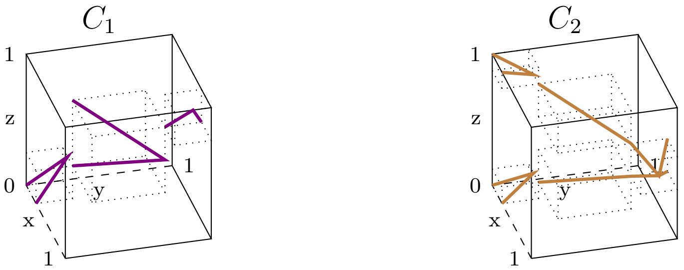

Figure 9: Supports of C_1 and C_2 in Example 21

$$C_2=\frac{1}{2}\ast_{M_3}\left(C_{e,\alpha},C_{e,\beta},C_{e,\gamma_2}\right)+\frac{1}{2}\ast_{\mathcal{A}}\left(C_{e,\alpha},C_{e,\beta},C_{e,\gamma_2}\right),$$

where $M_3(x,y,z)=\min\{x,y,z\}$ is the 3-dimensional comonotonic copula, $\mathcal{A}=(A_t)_{t\in\mathbb{I}}$, $A_t=D_1\mathbb{1}_{\left(0,\frac{1}{4}\right)}(t)+D_2\mathbb{1}_{\left[\frac{1}{4},\frac{3}{4}\right]}(t)+D_3\mathbb{1}_{\left(\frac{3}{4},1\right)}(t)$ and $D_1,D_2,D_3\in\mathscr{C}_3$ are such that $D_1\left(\frac{1}{2},1,\frac{1}{2}\right)=0=D_3\left(1,\frac{1}{2},\frac{1}{2}\right)$ (the existence of D_1,D_3 can be found in [20]).

A d-dimensional analog of Theorem 19 can be proved by similar arguments using the characterization of multivariate implicit dependence copulas in [31].

Theorem 22 (Characterization of extreme multivariate IDCs)**.** *Let $\alpha_1,\alpha_2,\dots,\alpha_d\in\mathscr{T}$ and $\mathcal{A}:=(A_t)_{t\in\mathbb{I}}$ be a collection of d-copulas. Then $\mathop{\ast_{\mathcal{A}}}\limits_{i=1}^{d}\left(C_{e,\alpha_i}\right)$ is extreme if and only if A_t is extreme in $\mathcal{S}\left(\times_{i=1}^d D_{\alpha_i}(t)\right)$ for a.e. $t\in\mathbb{I}$.*

When $\mathcal{A} = (\Pi_d)_{t\in\mathbb{I}}$, it is not hard to see that Theorem 20 and 22 are compatible. This follows from Lemma 13 and the fact that Π_d is extreme on $\mathcal{S}\left(\bigtimes_{i=1}^{d} R_i\right)$ only in the case $R_i = \{0,1\}$ for all but at most one index $i \in \{1,\ldots,d\}$.

Acknowledgements

This research project is supported by the Second Century Fund (C2F), Chulalongkorn University. The authors would like to thank the referees and the editor for their suggestions and comments that improve the manuscript tremendously.

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