

# Extremality of Factorizable Copulas and Implicit Dependence Copulas

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## Abstract

Copulas of continuous random variables satisfying  $f(X) = g(Y)$  a.s. for some Borel functions  $f$  and  $g$ , where  $f(X)$  has continuous distribution, are called *implicit dependence copulas*, of which a special subclass is the class of *factorizable copulas*, i.e., copulas that can be written as the Markov product of a left invertible copula and a right invertible copula in that order. Implicit dependence copulas were recently shown to coincide with generalized Markov products of left and right invertible copulas, where the joining copulas are arbitrary. In this paper, we obtain characterizations of extreme factorizable copulas and prove that implicit dependence copulas whose joining copula is one of the Fréchet-Hoeffding bounds are extreme. This condition, however, is not necessary, which will be shown via some examples of extreme implicit dependence copulas.

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## 1 Introduction

In probabilistic modeling, copulas are used to describe dependence among random variables. Copulas are also of enormous interest in their own right. See Durante-Sempi [12], Nelsen [15] and references therein. For instance, complete dependence copulas, defined as copulas of continuous random variables  $X$  and  $Y$  for which one is almost surely (a.s.) a Borel function of the other, have been studied extensively [23, 25] and applied to certain theoretical problems, see e.g. Schreyer, Paulin and Trutschnig [22] and Trutschnig [26]. Recently, there started novel investigations [17, 18, 19, 20, 24] on implicit dependence copulas, defined as copulas of continuous random variables  $X, Y$  for which there exist Borel

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functions  $f, g$  satisfying  $f(X) = g(Y)$  a.s. and  $f(X)$  has continuous distribution. Clearly, it reduces to a complete dependence copula if  $f$  or  $g$  is the identity map. But, unlike complete dependence copulas, for a given pair of  $f$  and  $g$ , implicit dependence copulas of  $X, Y$  for which  $f(X) = g(Y)$  a.s. are not unique in general. Pruntechapat and Sumetkijakan [20] studied a much more tractable subclass of the collection of implicit dependence copulas, called *factorizable copulas*. A given pair of  $f$  and  $g$  gives a unique factorizable copulas of  $X, Y$  for which  $f(X) = g(Y)$  a.s. Using different approaches, Ansari and Rüschendorf [1] and Panyasakulwong, Santiwipanont, Sumetkijakan and Yanpaisan [18] showed that implicit dependence copulas can be written as a generalized Markov product of two complete dependence copulas with respect to a class of joining copulas.

The space  $\mathcal{C}$  of bivariate copulas is clearly convex and compact with respect to the weak operator topology. As such, the extreme points of  $\mathcal{C}$ , called *extreme copulas* and defined as copulas that cannot be written as a convex sum of two distinct copulas, have been investigated in several of its equivalent forms for over five decades, see e.g. Brown and Shiflett [4], Douglas [8], Hestir and Williams [13] and Lindenstrauss [14]. Note that complete dependence copulas are basic examples of extreme points of  $\mathcal{C}$ . However, extreme copulas do not necessarily describe complete dependence relationship between random variables. For instance, the extremality of copulas with symmetric hairpin support was shown by Seethoff and Shiflett [23] and the so-called line-in-square copula was shown to be extreme by Applegate, Reeds, Scheinberg, Shepp and Shor [2, Theorem 1]. It was noted that these two examples are not *factorizable* [20]. Moreover, Pruntechapat, Santiwipanont and Sumetkijakan [19] showed that any extreme copula is not *strictly factorizable* in the sense that it cannot be written as a Markov product  $C_{e,\alpha} * C_{\beta,e}$ , where  $\alpha, \beta \in \mathcal{T}$  are not bijections. However, there actually are extreme copulas which are strictly factorizable (see Examples 2 and 12). In fact, the proof in [19] used an incorrect property of one-sided-invertible copulas. Ergo, this manuscript started out to identify all strictly factorizable extreme copulas. Our main results are characterizations of extreme factorizable copulas (Theorem 9 and Theorem 15) and a simple sufficient condition under which an implicit dependence copula is extreme (Theorem 17). See Figure 8.

The manuscript is organized as follows. Necessary background is given in the next section. In section 3, we characterize the extremality of factorizable copulas using some adjustments from [19] and a characterization of implicit dependence copulas from [18]. The condition involves the injective parts of the corresponding measure-preserving transformations. Section 4 gives a necessary and sufficient condition under which a non-atomic extreme copula is factorizable. The extremality of implicit dependence copulas whose joining copula is either  $M$  or  $W$  is also proved in the same section. To illustrate the variability of the joining copulas, a few examples of extreme implicit dependence copulas are then discussed.

## 2 Background

Let us denote  $\mathbb{I} := [0, 1]$ ,  $\mathcal{B}$  := the Borel  $\sigma$ -algebra on  $\mathbb{I}$ ,  $\lambda$  := Lebesgue measure on  $(\mathbb{I}, \mathcal{B})$ ,  $L^1(\mathbb{I}, \mathcal{S})$  := the space of  $\lambda$ -integrable  $\mathcal{S}$ -measurable functions where  $\mathcal{S}$  is a sub- $\sigma$ -algebra of  $\mathcal{B}$  and  $L^1(\mathbb{I}) := L^1(\mathbb{I}, \mathcal{B})$ . A function  $C: \mathbb{I}^2 \rightarrow \mathbb{I}$  is called a *copula* if it is grounded, i.e.,  $C(0, y) = 0 = C(x, 0)$  for all  $x, y$ , has uniform marginals, i.e.,  $C(1, y) = y$  and  $C(x, 1) = x$  for all  $x, y$  and is 2-increasing:

$$V_C(B) := C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0$$

for every rectangle  $B := [x_1, x_2] \times [y_1, y_2] \subseteq \mathbb{I}^2$ . The  $C$ -volume  $V_C$  can be extended to a unique Borel probability measure  $\mu_C$  such that for any  $A \in \mathcal{B}$ ,  $\mu_C(A \times \mathbb{I}) = \mu_C(\mathbb{I} \times A) = \lambda(A)$ . Furthermore, its definition implies that every copula is non-decreasing in each variable and Lipschitz with respect to the  $\ell^1$ -norm on  $\mathbb{I}^2$ . Therefore, the first partial derivatives of a copula exist almost everywhere with values in  $[0, 1]$ . See [12, 15] for more fundamental properties of copulas.

A Borel transformation  $f$  on  $\mathbb{I}$  is said to be *measure-preserving* if  $\lambda(f^{-1}(B)) = \lambda(B)$  for every  $B \in \mathcal{B}$ . Denoting the class of copulas and the class of measure-preserving transformations respectively by  $\mathcal{C}$  and  $\mathcal{T}$ . It is well-known that every copula can be written as  $C_{f,g}: \mathbb{I}^2 \rightarrow \mathbb{I}$ , defined by

$$C_{f,g}(x, y) := \lambda\left(f^{-1}([0, x]) \cap g^{-1}([0, y])\right), \quad (2.1)$$

for some  $f, g \in \mathcal{T}$ . Conversely, it is straightforward to show that every  $C_{f,g}$  is a copula although the representation is not unique. Proofs of this characterization of copulas can be found in de Amo et al. [7] and Vitale [27]. It is an elementary fact that the inverse image of a Borel set under a Borel map is again Borel. However, this is not always true for direct images. The direct image of a Borel set under a Borel map is called *Souslin* or *analytic*, see Bogachev [3]. The collection  $S_X$  of all Souslin subsets in a Hausdorff space  $X$  is closed under countable union and intersection. Furthermore, it is closed under taking the direct image and the inverse image under a Borel map. Nevertheless,  $S_X$  is not closed under complement. The complement of an analytic set is called *coanalytic*. Clearly, every Borel set is both analytic and coanalytic.

Continuous random variables  $X$  and  $Y$  are said to be *completely dependent* if there exists a Borel function  $f$  such that  $Y = f(X)$  a.s. or  $X = f(Y)$  a.s.; and they are said to be *implicitly dependent* if there exist Borel functions  $f$  and  $g$  such that  $f(X) = g(Y)$  a.s. and  $f(X)$  has continuous distribution. If  $X$  and  $Y$  are completely/implicitly dependent continuous random variables, then their copula  $C_{(X,Y)}$  is called a *complete/implicit dependence copula*. Let us denote the class of complete dependence copulas by  $\mathcal{C}_{\text{CD}}$  and the class of implicit dependence copulas by  $\mathcal{C}_{\text{ID}}$ . It is well known [12, 25] that a copula  $C$  is complete dependence if and only if  $C = C_{e,\psi}$  or  $C = C_{\psi,e}$  for some  $\psi \in \mathcal{T}$  where  $e$  is the identity on  $\mathbb{I}$ . In fact,  $C_{(U,\psi(U))} = C_{e,\psi}$  and  $C_{(\psi(U),U)} = C_{\psi,e}$  whenever  $U \sim \mathcal{U}(0, 1)$ .

Let  $(X, \mu)$  and  $(Y, \nu)$  be probability spaces. A linear operator  $T: L^1(X) \rightarrow L^1(Y)$  is called a *Markov operator* if  $T$  is positive,  $T\mathbb{1}_X = \mathbb{1}_Y$ , and  $\int_Y T f d\lambda = \int_X f d\lambda$  for all  $f \in L^1(X)$ . By standard arguments in functional analysis, every Markov operator has unit norm and its adjoint operator on  $L^\infty(\mathbb{I})$  can be extended to a unique Markov operator, denoted by  $T^*$ . Notable members of the class  $\mathcal{M}$  of Markov operators on  $L^1(\mathbb{I})$  are  $T_\psi: f \mapsto f \circ \psi$ , where  $\psi \in \mathcal{T}$ , whose adjoint  $T_\psi^*$  maps  $f \circ \psi$  to  $f$ . So it is clear that  $T_\psi$  and  $T_\psi^*$  are left and right invertible, respectively.  $\mathcal{M}$  is in a one-to-one correspondence with  $\mathcal{C}$  via the mappings  $\Psi: \mathcal{M} \rightarrow \mathcal{C}$  and  $\Phi: \mathcal{C} \rightarrow \mathcal{M}$ , where

$$\Psi(T)(x, y) := \int_0^x [T\mathbb{1}_{[0,y]}](s) ds \quad \text{and} \quad [\Phi(C)f](x) := \frac{d}{dx} \int_0^1 \partial_2 C(x, t) f(t) dt.$$

The class  $\mathcal{M}$  with the composition  $\circ$  is isomorphic to  $\mathcal{C}$  with the *Markov product*  $*$  defined by  $C * D(x, y) := \int_0^1 \partial_2 C(x, t) \partial_1 D(t, y) dt$ . Moreover,  $\Psi(T_f) = C_{e,f}$  and  $\Psi(T_f^*) = C_{f,e}$  are left and right invertible, respectively, and  $C_{f,g} = C_{f,e} * C_{e,g}$  for  $f, g \in \mathcal{T}$ , i.e., every copula can be factored as a product of right and left invertible copulas. Furthermore, the Markov product gives rise to a binary operation on the class  $\mathcal{D}$  of doubly stochastic

measures by  $(\mu * \nu)(E \times F) = \int_0^1 \frac{d}{dt} \mu(E \times [0, t]) \frac{d}{dt} \nu([0, t] \times F) dt$  for  $\mu, \nu \in \mathcal{D}$ . The *generalized Markov product* of copulas  $C$  and  $D$  with respect to a parametric class of *joining copulas*  $\mathcal{A} := \{A_t\}_{t \in [0,1]}$ , is formally defined as

$$C *_{\mathcal{A}} D(x, y) := \int_0^1 A_t(\partial_2 C(x, t), \partial_1 D(t, y)) dt \quad \text{for } x, y \in [0, 1].$$

Note that if the map  $(t, x, y) \mapsto A_t(x, y)$  is Borel measurable, then the function  $C *_{\mathcal{A}} D$  is a copula [10, 11, 12, 21]. The product is simply written  $C *_{\mathcal{A}} D$  if  $A_t = A$  for all (a.e.)  $t$ .

For a given  $C \in \mathcal{C}$  or its corresponding Markov operator  $T$ , the relation  $\Sigma_C = \Sigma_T$  contains all  $(R, S) \in \mathcal{B} \times \mathcal{B}$  for which  $T_C \mathbb{1}_S = \mathbb{1}_R$  a.e. The Borel sub- $\sigma$ -algebras  $\sigma_C^* = \sigma_T^*$  and  $\sigma_C = \sigma_T$  are defined as  $\pi_1(\Sigma_C)$  and  $\pi_2(\Sigma_C)$ , respectively, where  $\pi_i$  denotes the canonical projection on the  $i^{\text{th}}$ -axis. The copula  $C$  (and the corresponding Markov operator  $T_C$ ) is called *non-atomic* if both  $\sigma_C^*$  and  $\sigma_C$  are *non-atomic* in the sense that for each  $S \in \mathcal{S} = \sigma_C^*$  or  $\sigma_C$  satisfying  $\lambda(S) > 0$ , there is a subset  $R \in \mathcal{S}$  of  $S$  with Lebesgue measure strictly between 0 and  $\lambda(S)$ . In fact, it is known, see Sumetkijakan [24], that the atomicity of  $\sigma_C^*$  and  $\sigma_C$  always agree. Furthermore, by Darsow and Olsen [6], the non-atomicity of  $C$  implies that there exist  $f, g \in \mathcal{T}$  satisfying

$$\sigma_C^* \approx f^{-1}(\mathcal{B}) \quad \text{and} \quad \sigma_C \approx g^{-1}(\mathcal{B}). \quad (2.2)$$

For sub- $\sigma$ -algebras  $\mathcal{R}$  and  $\mathcal{S}$  of  $\mathcal{B}$ ,  $\mathcal{R} \approx \mathcal{S}$  means that for each  $R \in \mathcal{R}$  there is  $S \in \mathcal{S}$  such that  $\lambda(R \Delta S) = 0$  and for each  $S \in \mathcal{S}$  there is  $R \in \mathcal{R}$  such that  $\lambda(R \Delta S) = 0$ , where  $R \Delta S := (R \setminus S) \cup (S \setminus R)$ . The choice of  $f, g \in \mathcal{T}$  in (2.2) is unique up to a composition of an invertible measure-preserving transformation, i.e., if  $\sigma_C^* \approx f_1^{-1}(\mathcal{B})$  and  $\sigma_C \approx g_1^{-1}(\mathcal{B})$ , then there are  $h_1, h_2 \in \mathcal{I}_{\text{inv}} := \{h \in \mathcal{T} \mid \exists k \in \mathcal{T}, h \circ k = e = k \circ h \text{ a.e.}\}$  such that  $f_1 = h_1 \circ f$  a.e. and  $g_1 = h_2 \circ g$  a.e. In addition, if (2.2) holds then  $S := C_{f,e} * C * C_{e,g}$  is invertible and  $C' := C_{e,f} * S * C_{g,e}$  satisfies  $\sigma_{C'} = \sigma_C$  and  $\sigma_{C'}^* = \sigma_C^*$ . Note that  $C'$ , called the *isoalgebra factorizable copula* of  $C$ , is independent of the choice of  $f, g$  [20]. Finally, the condition  $C' = C$  is a criterion for verifying that  $C$  is *factorizable*, that is  $C = C_{e,\varphi} * C_{\psi,e}$  for some  $\varphi, \psi \in \mathcal{T}$ . For more details, see [20, 24].

### 3 Characterization of extreme factorizable copulas

It was shown in [19] that if an extreme copula is factorizable then it has to be one-sided invertible. However, there is an inaccurate point in the proof of this statement which also hints on how to obtain a counterexample. To see this, let us recall some basic facts on ordinal sums of copulas.

**Definition 1** ([12, 15]). Let  $\Lambda$  be a countable subset of  $\mathbb{N}$  (finite or countably infinite),  $\{(a_k, b_k)\}_{k \in \Lambda}$  a family of non-overlapping open intervals in  $[0, 1]$  and  $\{C_k\}_{k \in \Lambda}$  a family of copulas. Moreover, denote  $s_k(z) := \frac{z - a_k}{b_k - a_k}$  for  $z \in (a_k, b_k)$  and  $k \in \Lambda$ . Define the *ordinal sum*  $C := \langle a_k, b_k, C_k \rangle_{k \in \Lambda}$  of  $\{C_k\}_{k \in \Lambda}$  with respect to  $\{(a_k, b_k)\}_{k \in \Lambda}$  by

$$C(x, y) = \begin{cases} a_k + (b_k - a_k) C_k(s_k(x), s_k(y)) & \text{if } (x, y) \in (a_k, b_k)^2; \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Note that ordinal sums can be characterized as copulas  $C$  for which  $C(t, t) = t$  for some  $t \in (0, 1)$ . Moreover,  $C(t, t) = t$  if and only if  $C$  is the ordinal sum of two copulas with respect to  $\{(0, t), (t, 1)\}$  [15].

**Example 2.** Let  $\varphi, \psi \in \mathcal{T}$  be defined as follows:

$$\varphi(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{4}; \\ 2x - \frac{1}{2} & \text{if } \frac{1}{4} \leq x < \frac{1}{2}; \\ x & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2}; \\ 2x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x < \frac{3}{4}; \\ 2x - 1 & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases}$$

To show that the copula  $C := C_{e,\varphi} * C_{\psi,e}$ , whose graph is shown in Figure 1, is extreme,

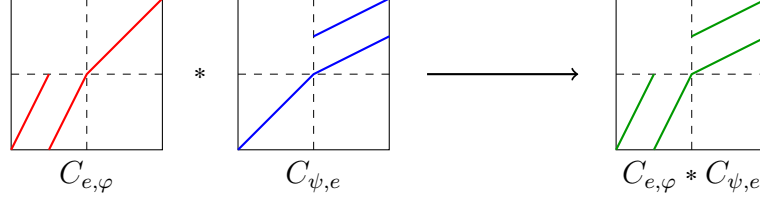


Figure 1: Supports of  $C_{e,\varphi}, C_{\psi,e}$  and their product for the counterexample

suppose  $C = kA + (1 - k)B$  for some  $k \in (0, 1)$  and  $A, B \in \mathcal{C}$ . As  $C$  is the product of two ordinal sums with respect to the same partition  $\{(0, \frac{1}{2}), (\frac{1}{2}, 1)\}$ , it follows that  $C(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ . Since the Fréchet-Hoeffding bounds for all copulas at  $(\frac{1}{2}, \frac{1}{2})$  are  $W(\frac{1}{2}, \frac{1}{2}) = 0$  and  $M(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ , this implies that  $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} = B(\frac{1}{2}, \frac{1}{2})$ . So  $C, A$  and  $B$  are ordinal sums of  $\{C_1, C_2\}, \{A_1, A_2\}$  and  $\{B_1, B_2\}$  with respect to  $\{(0, \frac{1}{2}), (\frac{1}{2}, 1)\}$ . By direct calculations (see Figure 1),  $C_1$  and  $C_2$  are left and right invertible, hence extreme. Since  $C_i = kA_i + (1 - k)B_i, i = 1, 2$ , we have  $C_1 = A_1 = B_1$  and  $C_2 = A_2 = B_2$  which imply that  $A = B = C$ . Therefore,  $C$  is an extreme point of  $\mathcal{C}$ .

Example 2 gives a pair  $\varphi, \psi \in \mathcal{T}$  for which the copula  $C_{e,\varphi} * C_{\psi,e}$  is extreme. Observe that  $\varphi^{-1}(\{\varphi(x)\})$  is a singleton for  $x \in (\frac{1}{2}, 1)$  and  $\psi^{-1}(\{\psi(x)\})$  is a singleton for  $x \in (0, \frac{1}{2})$ . Furthermore,  $\varphi([[\frac{1}{2}, 1]]) \cup \psi([0, \frac{1}{2}]) = [0, 1]$ . Notice that the proof of Theorem 5 in [19] hinges on a false claim [19, Lemma 4]: if  $\psi \in \mathcal{T}$  is not invertible and  $c \in (0, 1)$  then there exists a bounded  $\beta \geq 0$  such that  $T_\psi^* \beta = c\mathbb{1}$  and  $T_\psi^* ((\beta - c)\mathbb{1}_{[0,s]}) < 0$  a.e. for some  $s \in [0, 1]$ . Therefore, in order to revise [19, Theorem 5], we need more insights on the Markov operator  $T_\psi^*$ , where  $\psi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$ , which leads us to the following lemma.

**Lemma 3.** *Let  $\psi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$ . Then for any  $c \in (0, 1)$ , there is a non-constant non-negative function  $\beta \in L^\infty(\mathbb{I})$  such that  $T_\psi^* \beta = c\mathbb{1}$ . Also, let  $P_\beta := \{y \in \mathbb{I} : \beta(y) > c\}$  and  $N_\beta := \{y \in \mathbb{I} : \beta(y) < c\}$ . Then we obtain the following.*

1.  $T_\psi^* ((\beta - c)\mathbb{1}_A) = 0$  a.e. on  $\mathbb{I} \setminus \psi(A)$  for all  $A \in \mathcal{B}$ .
2. Both  $P_\beta$  and  $N_\beta$  have positive Lebesgue measure and for all Borel subsets  $C$  and  $D$  of  $P_\beta$  and  $N_\beta$ , respectively, with positive measure,

$$T_\psi^* ((\beta - c)\mathbb{1}_C) > 0 \quad \text{a.e. on } \psi(C) \quad \text{and} \quad (3.1)$$

$$T_\psi^* ((\beta - c)\mathbb{1}_D) < 0 \quad \text{a.e. on } \psi(D). \quad (3.2)$$

3. If  $C_\beta \subseteq P_\beta$  and  $D_\beta \subseteq N_\beta$  are Borel sets for which  $\lambda(C_\beta) = \lambda(P_\beta)$  and  $\lambda(D_\beta) = \lambda(N_\beta)$ , then  $\lambda(\psi(C_\beta) \Delta \psi(D_\beta)) = 0$  where  $A \Delta B := (A \setminus B) \cup (B \setminus A)$ .

*Proof.* Since  $\psi \in \mathcal{T}$  is not invertible, the linear operator  $T_\psi^*$  is not injective, i.e.,  $\ker T_\psi^* \neq \{0\}$ . So by the denseness of  $L^\infty(\mathbb{I})$  in  $L^1(\mathbb{I})$ , there exists  $\alpha \neq 0$  in  $L^\infty(\mathbb{I})$  such that  $T_\psi^* \alpha \equiv 0$  and hence there is  $k > 0$  for which  $k|\alpha(y)| \leq c$  for a.e.  $y \in \mathbb{I}$ . Let  $\beta := k\alpha + c\mathbb{1}$ . Then  $\beta$  satisfies the following properties and the first part of the statement is proved.

- $\beta(y) \in [0, 2c]$  a.e.
- $T_\psi^* \beta = kT_\psi^* \alpha + cT_\psi^* \mathbb{1} = c\mathbb{1}$ .
- $\beta$  is not a constant function. Otherwise,  $\alpha$  would be a nonzero constant function in the kernel of  $T_\psi^*$  which is impossible.

1.) Let  $A \in \mathcal{B}$ . Since  $\psi$  is a Borel map, the measurability of  $\psi(A)$  follows from [3, Theorem 6.7.3]. Then for  $B \in \mathcal{B}$  with  $\lambda(B \cap \psi(A)) = 0$  and a Borel set  $D$  containing  $\psi(A)$ , if  $\lambda(D) = \lambda(\psi(A))$  then  $\lambda(B \cap \psi(A)) = \lambda(B \cap D) = \lambda(\psi^{-1}(B) \cap \psi^{-1}(D)) \geq \lambda(\psi^{-1}(B) \cap A)$ . So  $\lambda(\psi^{-1}(B) \cap A) = 0$  and

$$\int_B T_\psi^* ((\beta - c) \mathbb{1}_A) d\lambda = \int_{\mathbb{I}} (\beta - c) \mathbb{1}_A \cdot T_\psi \mathbb{1}_B d\lambda = \int_{\mathbb{I}} (\beta - c) \mathbb{1}_{A \cap \psi^{-1}(B)} d\lambda = 0.$$

Since  $B$  is an arbitrary subset of  $\mathbb{I} \setminus \psi(A)$ ,  $T_\psi^* ((\beta - c) \mathbb{1}_A) = 0$  a.e. on  $\mathbb{I} \setminus \psi(A)$ .

2.) Since  $\beta \neq c\mathbb{1}$ , at least one of  $P_\beta$  and  $N_\beta$  must have positive measure and we may assume without loss of generality that  $\lambda(P_\beta) > 0$ . Let  $C$  be a positive measure Borel subset of  $P_\beta$ . So  $T_\psi^* ((\beta - c) \mathbb{1}_C) \geq 0$ . Let  $B$  be a positive-measure Borel subset of  $\psi(C)$  satisfying  $\lambda(C \cap \psi^{-1}(B)) = 0$ . The expression  $C \subseteq \psi^{-1}(\psi(C)) = \psi^{-1}(B) \dot{\cup} \psi^{-1}(\psi(C) \setminus B)$  implies that  $C \subseteq \psi^{-1}(\psi(C) \setminus B)$  a.e. So  $B \subseteq \psi(C) \subseteq \psi(\psi^{-1}(\psi(C) \setminus B)) = \psi(C) \setminus B$  a.e., which is a contradiction. Hence,  $\lambda(C \cap \psi^{-1}(B)) > 0$  and

$$\int_B T_\psi^* ((\beta - c) \mathbb{1}_C) d\lambda = \int_{\psi^{-1}(B)} (\beta - c) \mathbb{1}_C d\lambda = \int_{C \cap \psi^{-1}(B)} (\beta - c) d\lambda > 0,$$

which yields (3.1). Next, if  $\lambda(N_\beta) = 0$ , then

$$\mathbb{1} = \mathbb{1}_{P_\beta} + \mathbb{1}_{Z_\beta} + \mathbb{1}_{N_\beta} = \mathbb{1}_{C_\beta} + \mathbb{1}_{P_\beta \setminus C_\beta} + \mathbb{1}_{Z_\beta} = \mathbb{1}_{C_\beta} + \mathbb{1}_{Z_\beta} \text{ a.e.},$$

where  $Z_\beta := \{y \in \mathbb{I} : \beta(y) = c\}$  and  $C_\beta$  is a Borel subset of  $P_\beta$  with  $\lambda(C_\beta) = \lambda(P_\beta)$ . Consequently,

$$0 = T_\psi^* ((\beta - c) \mathbb{1}) = T_\psi^* ((\beta - c) \mathbb{1}_{C_\beta}) + T_\psi^* ((\beta - c) \mathbb{1}_{Z_\beta}) = T_\psi^* ((\beta - c) \mathbb{1}_{C_\beta}) \text{ a.e.},$$

which contradicts (3.1). So  $\lambda(N_\beta) > 0$  and (3.2) can be shown in a similar manner.

3.) Let  $A := \psi(C_\beta) \setminus \psi(D_\beta)$ . Then  $\mathbb{1} = \mathbb{1}_{C_\beta} + \mathbb{1}_{Z_\beta} + \mathbb{1}_{D_\beta}$  a.e. and

$$\begin{aligned} 0 &= \int_A T_\psi^* ((\beta - c) \mathbb{1}_{C_\beta}) d\lambda + \int_A T_\psi^* ((\beta - c) \mathbb{1}_{Z_\beta}) d\lambda + \int_A T_\psi^* ((\beta - c) \mathbb{1}_{D_\beta}) d\lambda \\ &= \int_A T_\psi^* ((\beta - c) \mathbb{1}_{C_\beta}) d\lambda, \end{aligned}$$

where the last equality follows from 1. If  $\lambda(A)$  were positive then (3.1) would imply that the last integral is positive, a contradiction. One can show that  $\lambda(\psi(D_\beta) \setminus \psi(C_\beta)) = 0$  in a similar way. Consequently,  $\lambda(\psi(C_\beta) \Delta \psi(D_\beta)) = 0$ .  $\square$

For a fixed  $\psi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$  and  $\beta \neq c\mathbb{1}$  satisfying  $T_\psi^* \beta = c\mathbb{1}$  a.e., if  $A$  is a positive-measure Borel set for which  $\beta > c$  on  $A$ , then, by Lemma 3(2),  $T_\psi^* ((\beta - c) \mathbb{1}_A) > 0$  a.e. on  $\psi(A)$ . Since  $T_\psi^* \beta = c\mathbb{1}$  a.e., the inequality implies that  $T_\psi^* ((\beta - c) \mathbb{1}_{\mathbb{I} \setminus A}) < 0$  a.e. on  $\psi(A)$ . By Lemma 3(1),  $T_\psi^* ((\beta - c) \mathbb{1}_{\mathbb{I} \setminus A}) = 0$  a.e. on  $\mathbb{I} \setminus \psi(\mathbb{I} \setminus A)$ . Hence,  $\psi(A) \cap (\mathbb{I} \setminus \psi(\mathbb{I} \setminus A))$  must

have measure zero, i.e. almost every element in  $\psi(A)$  is also in  $\psi(\mathbb{I} \setminus A)$ . Letting  $\mathbb{I}_\psi$  denote the image of the injective part for  $\psi$ :

$$\mathbb{I}_\psi := \{x \in \mathbb{I} : |\psi^{-1}(x)| = 1\},$$

it follows that  $\lambda(\psi(A) \cap \mathbb{I}_\psi) = 0$ . The next statement shows that  $\lambda(\psi(A) \cup \mathbb{I}_\psi) = 1$  for a suitably chosen  $\beta$ .

**Lemma 4.** *Let  $\psi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$  and  $c \in (0, 1)$ . Then there is a  $\beta_0$  in*

$$\mathcal{F}_\psi := \{\beta \in L^\infty(\mathbb{I}) : \beta(\mathbb{I}) \subseteq \mathbb{I} \text{ a.e. and } T_\psi^* \beta = c\mathbb{1} \text{ a.e.}\}$$

and a Borel subset  $C_{\beta_0}$  of  $P_{\beta_0} := \{y \in \mathbb{I} : \beta_0(y) > c\}$  such that  $\lambda(C_{\beta_0}) = \lambda(P_{\beta_0})$  and  $\lambda(\psi(C_{\beta_0}) \cup \mathbb{I}_\psi) = 1$ .

*Proof.* Observe that  $\mathcal{F}_\psi$  is a closed subset of  $S^* := \{f \in L^\infty(\mathbb{I}) : \|f\|_{L^\infty(\mathbb{I})} \leq 1\}$ , which is compact (in the weak\*-topology) by Banach-Alaoglu Theorem. So  $\mathcal{F}_\psi$  is also compact. It is easy to see that  $\mathcal{F}_\psi$  is a convex set containing  $c\mathbb{1}$ . Then Krein-Milman Theorem implies that there is an extreme point  $\beta_0 \in \mathcal{F}_\psi$ . Note that we have the following information.

- $\beta_0 \not\equiv c\mathbb{1}$  because  $c\mathbb{1}$  is not an extreme point.
- By (3.1) in Lemma 3(2), there exists a Borel set  $C_{\beta_0} \subseteq P_{\beta_0}$  such that  $\lambda(C_{\beta_0}) = \lambda(P_{\beta_0}) > 0$  and

$$T_\psi^* \left( (\beta_0 - c) \mathbb{1}_{C_{\beta_0}} \right) > 0 \quad \text{a.e. on } \psi(C_{\beta_0}).$$

Moreover, if we denote  $N_{\beta_0} := \{y \in \mathbb{I} : \beta_0(y) < c\}$  and  $D_{\beta_0}$  a Borel subset of  $N_{\beta_0}$  with  $\lambda(D_{\beta_0}) = \lambda(N_{\beta_0})$ , then Lemma 3(3) forces  $\lambda(\psi(C_{\beta_0}) \Delta \psi(D_{\beta_0})) = 0$  and, by (3.2) in Lemma 3(2),

$$T_\psi^* \left( (\beta_0 - c) \mathbb{1}_{D_{\beta_0}} \right) < 0 \quad \text{a.e. on } \psi(C_{\beta_0}).$$

If  $\lambda(\psi(C_{\beta_0}) \cup \mathbb{I}_\psi) < 1$ , then there exists a Borel subset  $E$  of  $\mathbb{I} \setminus (\psi(C_{\beta_0}) \cup \mathbb{I}_\psi)$  with the same measure. Since  $\lambda(\psi(C_{\beta_0}) \Delta \psi(D_{\beta_0})) = 0$ ,  $\beta_0(y) = c$  for a.e.  $y \in \psi^{-1}(E)$ . Moreover,  $|\psi^{-1}(x)| \geq 2$  for a.e.  $x \in E$ , or  $\psi$  is not injective on  $\psi^{-1}(E)$ . Then there exists  $\rho \in \ker(T_\psi^*) \setminus \{0\}$  such that  $\text{supp}(\rho) \subseteq \psi^{-1}(E)$  and  $|\rho(y)| \leq \min\{1 - c, c\}$  for a.e.  $y \in \mathbb{I}$ . In fact,  $\varphi := \psi|_{\psi^{-1}(E)}$  is a measure-preserving transformation from  $\psi^{-1}(E)$  to  $E$  and hence  $T_\varphi^* : L^\infty(\psi^{-1}(E)) \rightarrow L^\infty(E)$  is a Markov operator. Then there exists  $\xi \in \ker(T_\varphi^*) \setminus \{0\}$  such that  $|\xi(y)| \leq \min\{1 - c, c\}$  for a.e.  $y \in \psi^{-1}(E)$ . Thus,  $\rho := \xi \cdot \mathbb{1}_{\psi^{-1}(E)} \in L^\infty(\mathbb{I})$  satisfies the desired properties. Let  $\beta_1 := \beta_0 + \rho$  and  $\beta_2 := \beta_0 - \rho$ . Then both  $\beta_1$  and  $\beta_2$  are in  $\mathcal{F}_\psi$  and not equal to  $\beta_0 = \frac{1}{2}(\beta_1 + \beta_2)$ , which contradicts the extremality of  $\beta_0$ . Thus,  $\lambda(\psi(C_{\beta_0}) \cup \mathbb{I}_\psi) = 1$ .  $\square$

By adjusting the proof of Theorem 5 in [19] using Lemma 3 and 4, we obtain an accurate version as stated in the next theorem. Certain details in the proof that mimic the corresponding arguments in [19, Theorem 5]

**Theorem 5.** *If a Markov operator is of the form  $T_\varphi \circ T_\psi^*$  for some  $\varphi, \psi \in \mathcal{T}$  satisfying  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\psi) < 1$ , then it cannot be extreme.*

*Proof.* Suppose that  $T = T_\varphi \circ T_\psi^*$  where  $\varphi, \psi \in \mathcal{T}$  and  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\psi) < 1$ . Then neither  $\varphi$  nor  $\psi$  is invertible as  $\lambda(\mathbb{I}_\xi) = 1$  whenever  $\xi \in \mathcal{T}_{\text{inv}}$ . Let  $\mu, \mu_1$  and  $\mu_2$  be the doubly stochastic measures associated with  $T, T_\varphi$  and  $T_\psi^*$ , respectively. So  $\mu = \mu_1 * \mu_2$ . By the same proof as in [19, Theorem 5], Lemma 4 yields measures  $\mu_{11}$  and  $\mu_{21}$  defined on  $\mathcal{B}(\mathbb{I}^2)$  by

$$\mu_{11}(E) := \int_E \beta(s) d\mu_1(s, t) = \int_0^1 \beta(s) \mathbb{1}_{E^s}(\varphi(s)) d\lambda(s) \quad \text{and} \quad (3.3)$$

$$\mu_{21}(E) := \int_E \gamma(t) d\mu_2(s, t) = \int_0^1 \gamma(t) \mathbb{1}_{E^t}(\psi(t)) d\lambda(t) \quad (3.4)$$

where  $\beta, \gamma \in L^\infty(\mathbb{I})$  satisfy  $T_\varphi^* \beta = T_\psi^* \gamma = \frac{1}{2} \mathbb{1}$  and  $E_x, E^y$  denote respectively the  $x$ -section and the  $y$ -section of  $E$ . Still following the proof of [19, Theorem 5], if we define measures  $\mu_{12}(E) := \mu_1(E) - \mu_{11}(E)$  and  $\mu_{22}(E) := \mu_2(E) - \mu_{21}(E)$  for  $E \in \mathcal{B}(\mathbb{I}^2)$ , then  $\mu_3 := 2(\mu_{11} * \mu_{21} + \mu_{12} * \mu_{22})$  and  $\mu_4 := 2(\mu_{11} * \mu_{22} + \mu_{12} * \mu_{21})$  are doubly stochastic measures such that  $\mu = \frac{\mu_3 + \mu_4}{2}$ .

To show that  $\mu$ , i.e.  $T$ , is extreme, it remains to verify that  $\mu_3 \neq \mu_4$ . This last part of the proof relies heavily on the two newly proven lemmas. By Lemma 3(2) and Lemma 4, the chosen functions  $\beta$  and  $\gamma$  also satisfy  $M_\varphi := T_\varphi^* \left( (\beta - \frac{1}{2}) \mathbb{1}_{C_\beta} \right) > 0$  on  $\varphi(C_\beta)$  and  $M_\psi := T_\psi^* \left( (\gamma - \frac{1}{2}) \mathbb{1}_{C_\gamma} \right) > 0$  on  $\psi(C_\gamma)$ , where

$$\lambda(\varphi(C_\beta) \cup \mathbb{I}_\varphi) = \lambda(\psi(C_\gamma) \cup \mathbb{I}_\psi) = 1. \quad (3.5)$$

Here,  $C_\beta$  (resp.,  $C_\gamma$ ) is a Borel subset of  $P_\beta := \{y \in \mathbb{I} : \beta(y) > c\}$  (resp.,  $P_\gamma$ ) with the same measure. Applying  $\mu_3 - \mu_4$  to  $E := C_\beta \times C_\gamma$  gives

$$\begin{aligned} (\mu_3 - \mu_4)(E) &= 2[(\mu_{11} - \mu_{12}) * (\mu_{21} - \mu_{22})](E) \\ &= 8 \left[ \left( \mu_{11} - \frac{1}{2} \mu_1 \right) * \left( \mu_{21} - \frac{1}{2} \mu_2 \right) \right](E) \\ &= 8 \int_0^1 \frac{d}{dz} \left( \mu_{11} - \frac{1}{2} \mu_1 \right) (C_\beta \times [0, z]) \cdot \frac{d}{dz} \left( \mu_{21} - \frac{1}{2} \mu_2 \right) ([0, z] \times C_\gamma) d\lambda(z). \end{aligned}$$

Now, the expression (3.3) gives

$$\begin{aligned} \frac{d}{dz} \left( \mu_{11} - \frac{1}{2} \mu_1 \right) (C_\beta \times [0, z]) &= \frac{d}{dz} \int_{C_\beta} \left( \beta(s) - \frac{1}{2} \right) \mathbb{1}_{[0, z]}(\varphi(s)) d\lambda(s) \\ &= \frac{d}{dz} \int_0^1 \mathbb{1}_{C_\beta}(s) \left( \beta(s) - \frac{1}{2} \right) \partial_2 C_{\varphi, e}(z, s) d\lambda(s) = M_\varphi(z). \end{aligned}$$

Similarly, (3.4) gives  $\frac{d}{dz} \left( \mu_{21} - \frac{1}{2} \mu_2 \right) ([0, z] \times C_\gamma) = M_\psi(z)$ . So

$$\begin{aligned} (\mu_3 - \mu_4)(E) &= 8 \int_0^1 M_\varphi \cdot M_\psi d\lambda \\ &= 8 \int_{\mathbb{I}_\varphi \cup \mathbb{I}_\psi} M_\varphi \cdot M_\psi d\lambda + 8 \int_{\mathbb{I} \setminus (\mathbb{I}_\varphi \cup \mathbb{I}_\psi)} M_\varphi \cdot M_\psi d\lambda \\ &= 8 \int_{\varphi(C_\beta) \cap \psi(C_\gamma)} M_\varphi \cdot M_\psi d\lambda > 0. \end{aligned} \quad (3.6)$$

The last equality is a consequence of the following. Note from the discussion after Lemma 3 that

$$\lambda(\varphi(C_\beta) \cap \mathbb{I}_\varphi) = \lambda(\psi(C_\gamma) \cap \mathbb{I}_\psi) = 0. \quad (3.7)$$

- By (3.7) and Lemma 3(1),  $M_\varphi \equiv 0$  a.e. on  $\mathbb{I}_\varphi \subseteq \mathbb{I} \setminus \varphi(C_\beta)$  and  $M_\psi \equiv 0$  a.e. on  $\mathbb{I}_\psi \subseteq \mathbb{I} \setminus \psi(C_\gamma)$ . So  $\int_{\mathbb{I}_\varphi \cup \mathbb{I}_\psi} M_\varphi \cdot M_\psi \, d\lambda = 0$ .
- (3.7) also implies that

$$\begin{aligned} \lambda((\varphi(C_\beta) \cap \psi(C_\gamma)) \cap (\mathbb{I}_\varphi \cup \mathbb{I}_\psi)) &\leq \lambda((\varphi(C_\beta) \cap \mathbb{I}_\varphi) \cup (\psi(C_\gamma) \cap \mathbb{I}_\psi)) \\ &\leq \lambda(\varphi(C_\beta) \cap \mathbb{I}_\varphi) + \lambda(\psi(C_\gamma) \cap \mathbb{I}_\psi) = 0. \end{aligned}$$

On the other hand, (3.5) gives

$$\lambda((\varphi(C_\beta) \cap \psi(C_\gamma)) \cup (\mathbb{I}_\varphi \cup \mathbb{I}_\psi)) \geq \lambda((\varphi(C_\beta) \cup \mathbb{I}_\varphi) \cap (\psi(C_\gamma) \cup \mathbb{I}_\psi)) = 1.$$

Hence  $\lambda((\varphi(C_\beta) \cap \psi(C_\gamma)) \Delta (\mathbb{I} \setminus (\mathbb{I}_\varphi \cup \mathbb{I}_\psi))) = 0$ , i.e.,  $\int_{\mathbb{I} \setminus (\mathbb{I}_\varphi \cup \mathbb{I}_\psi)} M_\varphi \cdot M_\psi \, d\lambda = \int_{\varphi(C_\beta) \cap \psi(C_\gamma)} M_\varphi \cdot M_\psi \, d\lambda$  as desired.

Moreover, the inequality (3.6) comes from the properties of  $\beta$  and  $\gamma$  discussed above, including the assumption  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\psi) < 1$ . Thus,  $\mu_3 \neq \mu_4$ .  $\square$

The theorem above can be rephrased as: for any  $\varphi, \psi \in \mathcal{T}$ , if  $C_{e,\varphi} * C_{\psi,e}$  is extreme then  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\psi) = 1$ . To investigate the converse, we use a characterization of implicit dependence copulas in terms of generalized Markov product stated as follows.

**Theorem 6.** *Let  $\alpha, \beta \in \mathcal{T}$ .*

- *The generalized Markov product  $C := C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$  is an implicit dependence copula whenever the class of joining copulas  $\mathcal{A} = \{A_t\}_{t \in \mathbb{I}}$  makes such product well-defined. In other words, there exist  $X, Y \sim \mathcal{U}(0,1)$  such that  $\alpha(X) = \beta(Y)$  a.s. and  $C = C_{(X,Y)}$ .*
- *Conversely, if  $X, Y \sim \mathcal{U}(0,1)$  have  $C$  as their copula and satisfy  $\alpha(X) = \beta(Y)$  a.s. then there exists a class of joining copulas  $\mathcal{A} = \{A_t\}_{t \in \mathbb{I}}$  such that  $C = C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ .*

This theorem is investigated first by Panyasakulwong et al. [17] and is extended to a general version in [18], in which the formula for joining copulas  $A_t$  in the case that  $\alpha, \beta \in \mathcal{T}$  are countably piecewise monotonic surjections is derived. The complete version of this theorem is inspired by [1].

It is well-known [6] that for any  $\alpha \in \mathcal{T}$ ,  $\partial_1 C_{e,\alpha}(x, y) = \mathbb{1}_{[0,y]}(\alpha(x)) \in \{0, 1\}$  a.e. but  $\partial_2 C_{e,\alpha}(x, y)$  cannot be written in a similar form. However, the following statement shows that  $\partial_2 C_{e,\alpha}$  can be expressed explicitly on  $\mathbb{I} \times \mathbb{I}_\alpha$ .

**Lemma 7.** *Let  $\alpha \in \mathcal{T}$  and  $x \in \mathbb{I}$ . Then  $\partial_2 C_{e,\alpha}(x, t) = \partial_1 C_{\alpha,e}(t, x) \in \{0, 1\}$  for a.e.  $t \in \mathbb{I}_\alpha$ .*

*Proof.* Notice that for a.e.  $t \in \mathbb{I}_\alpha$ ,  $\partial_1 C_{\alpha,e}(t, x) = T_\alpha^* \mathbb{1}_{[0,x]}(t)$  by the correspondence between copulas and Markov operators [12]. Now, for every Borel subset  $B$  of  $\mathbb{I}_\alpha$ ,

$$\int_B T_\alpha^* \mathbb{1}_{[0,x]}(y) \, d\lambda(y) = \int_{\alpha^{-1}(B)} \mathbb{1}_{[0,x]}(z) \, d\lambda(z) = \int_B \mathbb{1}_{[0,x]} \circ \alpha^{-1}(s) \, d\lambda(s), \quad (3.8)$$

where the last equality follows from the change of variable, see Dudley [9, Theorem 4.1.11 & Problem 7 p. 122], under the assumption that  $\alpha^{-1}|_B$  is injective and measure-preserving. Since the expression (3.8) is true for all  $B \subseteq \mathbb{I}_\alpha$ , we have  $T_\alpha^* \mathbb{1}_{[0,x]}(t) = \mathbb{1}_{[0,x]} \circ \alpha^{-1}(t)$ , i.e.,  $\partial_1 C_{\alpha,e}(t, x) \in \{0, 1\}$  for a.e.  $t \in \mathbb{I}_\alpha$  as desired.  $\square$

Now, we are ready to show the converse direction of Theorem 5.

**Lemma 8.** *Let  $\varphi, \psi \in \mathcal{T}$  be such that  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\psi) = 1$ . If  $D \in \mathcal{C}$  is such that  $\mu_D$  is absolutely continuous with respect to  $\mu_{C_{e,\varphi} * C_{\psi,e}}$ , written  $\mu_D \ll \mu_{C_{e,\varphi} * C_{\psi,e}}$ , then  $D = C_{e,\varphi} * C_{\psi,e}$ .*

*Proof.* Let  $D \in \mathcal{C}$  be such that  $\mu_D \ll \mu_{C_{e,\varphi} * C_{\psi,e}}$ . By [24, Theorem 4.6],  $D \in \mathcal{C}_{\text{ID}}$  is a copula of  $X, Y \sim \mathcal{U}(0, 1)$  satisfying  $\varphi(X) = \psi(Y)$  a.s. and, by Theorem 6,  $D = C_{e,\varphi} *_{\mathcal{A}} C_{\psi,e}$  for some  $\mathcal{A} = \{A_t\}_{t \in \mathbb{I}} \subseteq \mathcal{C}$ . For fixed  $a, b, c, d \in \mathbb{I}$  such that  $a \leq b$  and  $c \leq d$ , let us consider the  $D$ -volume of  $R := [a, b] \times [c, d]$ . By a straightforward derivation,  $V_D(R) = \int_{\mathbb{I}} F(t) d\lambda(t)$  where  $F(t) := V_{A_t}([\partial_2 C_{e,\varphi}(a, t), \partial_2 C_{e,\varphi}(b, t)] \times [\partial_1 C_{\psi,e}(t, c), \partial_1 C_{\psi,e}(t, d)])$ .

If  $t \in \mathbb{I}_\varphi$ , then Lemma 7 yields that  $\partial_2 C_{e,\varphi}(a, t), \partial_2 C_{e,\varphi}(b, t) \in \{0, 1\}$  a.e.  $t$ , i.e.,  $[\partial_2 C_{e,\varphi}(a, t), \partial_2 C_{e,\varphi}(b, t)] \in \{\{0\}, \{1\}, \mathbb{I}\}$ , which implies that

$$F(t) = (\partial_2 C_{e,\varphi}(b, t) - \partial_2 C_{e,\varphi}(a, t)) (\partial_1 C_{\psi,e}(t, d) - \partial_1 C_{\psi,e}(t, c)).$$

If  $t \in \mathbb{I} \setminus \mathbb{I}_\varphi$  then  $t \in \mathbb{I}_\psi$ , for a.e.  $t$ , and so  $\partial_1 C_{\psi,e}(t, c), \partial_1 C_{\psi,e}(t, d) \in \{0, 1\}$ . Similarly to the above,  $F(t) = (\partial_2 C_{e,\varphi}(b, t) - \partial_2 C_{e,\varphi}(a, t)) (\partial_1 C_{\psi,e}(t, d) - \partial_1 C_{\psi,e}(t, c))$ . Therefore,

$$\begin{aligned} V_D(R) &= \int_{\mathbb{I}} F(t) d\lambda(t) = \int_{\mathbb{I}} (\partial_2 C_{e,\varphi}(b, t) - \partial_2 C_{e,\varphi}(a, t)) (\partial_1 C_{\psi,e}(t, d) - \partial_1 C_{\psi,e}(t, c)) d\lambda(t) \\ &= V_{C_{e,\varphi} * C_{\psi,e}}(R). \end{aligned}$$

This implies that  $D = C_{e,\varphi} * C_{\psi,e}$  as  $\mathcal{B}(\mathbb{I}^2)$  is generated by  $\{[a, b] \times [c, d] \subseteq \mathbb{I}^2\}$ .  $\square$

**Theorem 9.** *For  $\varphi, \psi \in \mathcal{T}$ ,  $C_{e,\varphi} * C_{\psi,e}$  is extreme if and only if  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\psi) = 1$ .*

*Proof.* Theorem 5 is the “only if” part. To prove the “if” part, let  $D := C_{e,\varphi} * C_{\psi,e}$  and  $\mu_1, \mu_2$  be doubly stochastic measures such that  $\mu_D = c\mu_1 + (1 - c)\mu_2$  for some  $c \in (0, 1)$ . Then  $\mu_1 \ll \mu_D$  and  $\mu_2 \ll \mu_D$ . Thus, by Lemma 8,  $\mu_1 = \mu_2 = \mu_D$ , i.e.,  $D = C_{e,\varphi} * C_{\psi,e}$  is extreme.  $\square$

For any pair  $(\varphi, \psi) \in \mathcal{T} \times \mathcal{T}$  satisfying  $\lambda(\mathbb{I}_\varphi) + \lambda(\mathbb{I}_\psi) < 1$  and  $s \in \mathcal{T}_{\text{inv}}$ , we have  $C_{e,\varphi} * C_{e,s} * C_{\psi,e} = C_{e,s \circ \varphi} * C_{\psi,e}$  where

$$\mathbb{I}_{s \circ \varphi} = \{x \in \mathbb{I} : |\varphi^{-1}(s^{-1}(x))| = 1\} = s(\mathbb{I}_\varphi), \quad (3.9)$$

i.e.,  $\lambda(\mathbb{I}_{s \circ \varphi}) = \lambda(s(\mathbb{I}_\varphi)) = \lambda(\mathbb{I}_\varphi)$ . Hence,  $\lambda(\mathbb{I}_{s \circ \varphi} \cup \mathbb{I}_\psi) \leq \lambda(\mathbb{I}_\varphi) + \lambda(\mathbb{I}_\psi) < 1$  which gives the following corollary.

**Corollary 10.** *For any  $\varphi, \psi \in \mathcal{T}$  with  $\lambda(\mathbb{I}_\varphi) + \lambda(\mathbb{I}_\psi) < 1$  and  $S \in \mathcal{C}_{\text{inv}}$ , the copula  $C_{e,\varphi} * S * C_{\psi,e}$  is not extreme.*

Observe that in the case  $\varphi = \psi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$ ,  $\lambda(\mathbb{I}_\varphi \cup \mathbb{I}_\varphi) = \lambda(\mathbb{I}_\varphi) < 1$ . So any idempotent non-atomic copula which is not  $M$  is not extreme.

**Corollary 11.** *For any  $\varphi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$ , the copula  $C_{e,\varphi} * C_{\varphi,e}$  is not extreme.*

From this corollary and the characterization of idempotent copulas [6] under the usual Markov product  $*$ , we can see that  $M$  is the only extreme idempotent copula.

Nonetheless, the Markov product  $C_{e,\varphi} * S * C_{\varphi,e}$  where  $\varphi \in \mathcal{T} \setminus \mathcal{T}_{\text{inv}}$  and  $S \in \mathcal{C}_{\text{inv}}$  may be extreme. The reason is that it can be written as  $C_{e,s \circ \varphi} * C_{\varphi,e}$ , where  $S = C_{e,s}$ , which is extreme if  $\lambda(\mathbb{I}_{s \circ \varphi} \cup \mathbb{I}_\varphi) = 1$ .

**Example 12.** Let  $\varphi \in \mathcal{T}$  and  $s_{\frac{1}{2}} \in \mathcal{T}_{\text{inv}}$  be defined as follows:

$$\varphi(x) := \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{4}; \\ 2x - \frac{1}{2} & \text{if } \frac{1}{4} \leq x < \frac{1}{2}; \\ x & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad s_{\frac{1}{2}}(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 \leq x < \frac{1}{2}; \\ x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

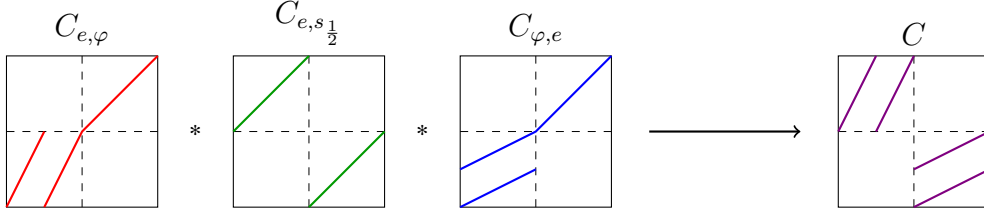


Figure 2: Supports of  $C$  and related copulas for the counterexample

The supports of  $C := C_{e,\varphi} * C_{e,s_{\frac{1}{2}}} * C_{\varphi,e}$  and related copulas are shown in Figure 2. Note that  $\mathbb{I}_\varphi = (\frac{1}{2}, 1]$  and  $\mathbb{I}_{s_{\frac{1}{2}} \circ \varphi} = [0, \frac{1}{2})$  whose union has measure 1. So Theorem 9 implies that  $C$  is extreme.

## 4 Extremality of implicit dependence copulas

In addition to characterizing extreme factorizable copulas, Theorem 6 will also be used to show the absolute continuity of non-atomic copula  $C$  with respect to the corresponding isoalgebra factorizable copula  $C'$ . Roughly speaking, the support of  $C$  is contained in the support of  $C'$ . To see this, we first need to verify the following result on the absolute continuity of the generalized and usual Markov products.

**Lemma 13.** *Let  $C, D \in \mathcal{C}$  and  $\mathcal{A} = \{A_t\}_{t \in \mathbb{I}}$  be a collection of bivariate copulas for which  $C *_{\mathcal{A}} D$  is well-defined. Then  $\mu_{C *_{\mathcal{A}} D}$  is absolutely continuous with respect to  $\mu_{C * D}$ .*

*Proof.* Since  $\mathcal{B}(\mathbb{I}^2) = \sigma(\{[a, b] \times [c, d] \subseteq \mathbb{I}^2\})$ , it suffices to show that  $V_{C * D}(R) = 0$  implies  $V_{C *_{\mathcal{A}} D}(R) = 0$  for all  $R := [a, b] \times [c, d] \subseteq \mathbb{I}^2$ . To this end, let  $a, b, c, d \in \mathbb{I}$  be such that  $a \leq b, c \leq d$  and  $V_{C * D}(R) = 0$  where  $R = [a, b] \times [c, d]$ . Recall that for any collection of bivariate copulas  $\mathcal{B} = \{B_t\}_{t \in \mathbb{I}}$ ,

$$V_{C *_{\mathcal{B}} D}(R) = \int_0^1 V_{B_t}([\partial_2 C(a, t), \partial_2 C(b, t)] \times [\partial_1 D(t, c), \partial_1 D(t, d)]) dt.$$

In particular,  $V_{C * D}(R) = \int_0^1 (\partial_2 C(b, t) - \partial_2 C(a, t)) (\partial_1 D(t, d) - \partial_1 D(t, c)) dt$ . Since  $\partial_2 C$  and  $\partial_1 D$  are increasing in the first and second coordinates, respectively, the assumption implies that  $\partial_2 C(a, t) = \partial_2 C(b, t)$  or  $\partial_1 D(t, c) = \partial_1 D(t, d)$  a.e.  $t \in \mathbb{I}$ . Hence, the  $A_t$ -volume of  $[\partial_2 C(a, t), \partial_2 C(b, t)] \times [\partial_1 D(t, c), \partial_1 D(t, d)]$  is zero for a.e.  $t$  and  $V_{C *_{\mathcal{A}} D}(R) = 0$ .  $\square$

Recall that if  $C$  is a non-atomic copula with  $\sigma_C^* \approx f^{-1}(\mathcal{B})$  and  $\sigma_C \approx g^{-1}(\mathcal{B})$  for some  $f, g \in \mathcal{T}$  then  $S_C := C_{f,e} * C * C_{e,g}$  is invertible and the *isoalgebra factorizable copula* of  $C$  is defined as  $C' := C_{e,f} * S_C * C_{g,e}$ .

**Corollary 14.** *Let  $C$  be a non-atomic copula with corresponding isoalgebra factorizable copula  $C'$ . Then  $\mu_C \ll \mu_{C'}$ .*

*Proof.* Let  $f, g \in \mathcal{T}$  be such that  $\sigma_C^* \approx f^{-1}(\mathcal{B})$  and  $\sigma_C \approx g^{-1}(\mathcal{B})$ . Then  $C' = C_{e,f} * S_C * C_{g,e}$  where  $S_C = C_{e,s}$  for some  $s \in \mathcal{T}_{\text{inv}}$  and  $\mu_C(\{(x, y) \in \mathbb{I}^2 : (s \circ f)(x) = g(y)\}) = 1$  by [20, 24]. The latter statement implies that  $C \in \mathcal{C}_{\text{ID}}$  and then by Theorem 6, there is  $\mathcal{A} \subseteq \mathcal{C}$  such that  $C = C_{e,s \circ f} *_{\mathcal{A}} C_{g,e}$ . Applying Lemma 13 to  $C$  yields.  $\mu_C = \mu_{C_{e,s \circ f} *_{\mathcal{A}} C_{g,e}} \ll \mu_{C_{e,s \circ f} * C_{g,e}} = \mu_{C_{e,f} * C_{e,s} * C_{g,e}} = \mu_{C'}$ .  $\square$

Theorem 9 gave a characterization of the extremality of copulas under a factorizable assumption. In the following theorem, the factorizability of extreme non-atomic copulas is characterized.

**Theorem 15.** *Let  $C$  be a non-atomic extreme copula with  $\sigma_C^* \approx f^{-1}(\mathcal{B})$  and  $\sigma_C \approx g^{-1}(\mathcal{B})$  for some  $f, g \in \mathcal{T}$ . Recall that  $S_C := C_{f,e} * C * C_{e,g} = C_{e,s}$  for some  $s \in \mathcal{T}_{\text{inv}}$ . Then  $C$  is factorizable if and only if  $s^{-1}(t) \in \mathbb{I}_f$  for almost every  $t \in \mathbb{I} \setminus \mathbb{I}_g$ .*

*Proof.* ( $\Rightarrow$ ) If  $C$  is factorizable, then  $C = C' := C_{e,f} * S_C * C_{g,e} = C_{e,s \circ f} * C_{g,e}$ . Since  $C$  is extreme, it follows from Theorem 9 that  $\lambda(\mathbb{I}_{s \circ f} \cup \mathbb{I}_g) = 1$ . As a consequence of (3.9), for almost every  $t \in \mathbb{I} \setminus \mathbb{I}_g$ ,  $t \in s(\mathbb{I}_f)$ , i.e.,  $s^{-1}(t) \in \mathbb{I}_f$  as desired.

( $\Leftarrow$ ) Suppose that  $s^{-1}(t) \in \mathbb{I}_f$  for almost all  $t \in \mathbb{I} \setminus \mathbb{I}_g$ , i.e.,  $\mathbb{I} \setminus \mathbb{I}_g \subseteq s(\mathbb{I}_f) = \mathbb{I}_{s \circ f}$  except possibly for points in a subset of measure zero. Then  $\lambda(\mathbb{I}_{s \circ f} \cup \mathbb{I}_g) \geq \lambda((\mathbb{I} \setminus \mathbb{I}_g) \cup \mathbb{I}_g) = 1$ , i.e.,  $\lambda(\mathbb{I}_{s \circ f} \cup \mathbb{I}_g) = 1$ . This implies that  $C_{e,s \circ f} * C_{g,e} = C_{e,f} * S_C * C_{g,e} = C'$  is extreme by Theorem 9. Since  $C$  is non-atomic, Corollary 14 and then Lemma 8 give  $C = C'$ , that is,  $C$  is factorizable.  $\square$

**Example 16.** Let  $C_1$  and  $C_2$  be extreme copulas whose probability masses concentrate on the supports shown in Figure 3.

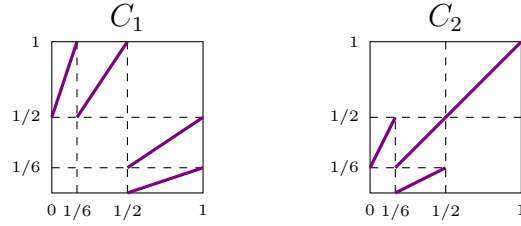


Figure 3: Supports of  $C_1$  and  $C_2$  in Example 16

Notice that the corresponding Markov operators are given by

$$T_{C_1}h(x) = \begin{cases} h(3x + \frac{1}{2}) & \text{if } 0 \leq x \leq \frac{1}{6}; \\ h(\frac{3}{2}x + \frac{1}{4}) & \text{if } \frac{1}{6} < x \leq \frac{1}{2}; \\ \frac{1}{3}h(\frac{1}{3}x - \frac{1}{6}) + \frac{2}{3}h(\frac{2}{3}x - \frac{1}{6}) & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \quad \text{and}$$

$$T_{C_2}h(x) = \begin{cases} h(2x + \frac{1}{6}) & \text{if } 0 \leq x \leq \frac{1}{6}; \\ \frac{1}{2}h(\frac{1}{2}x - \frac{1}{12}) + \frac{1}{2}h(x) & \text{if } \frac{1}{6} < x \leq \frac{1}{2}; \\ h(x) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

It is straightforward to show that  $\sigma_{C_1} = \sigma_{C_1}^* = \sigma_{C_2} = \sigma_{C_2}^*$  is generated by  $B_1 \cup (2B_1 + \frac{1}{6}) \cup B_2$ , where  $B_1 \in \mathcal{B}([0, \frac{1}{6}])$  and  $B_2 \in \mathcal{B}([\frac{1}{2}, 1])$ . This  $\sigma$ -algebra is essentially equivalent to  $f^{-1}(\mathcal{B})$  where  $f(x) = 3x\mathbb{1}_{[0, \frac{1}{6}]}(x) + (\frac{3}{2}x - \frac{1}{4})\mathbb{1}_{(\frac{1}{6}, \frac{1}{2}]}(x) + x\mathbb{1}_{(\frac{1}{2}, 1]}(x)$ . Furthermore, we can compute  $S_1 := C_{f,e} * C_1 * C_{e,f} = C_{e,s_{\frac{1}{2}}}$  where  $s_{\frac{1}{2}}$  is defined in Example 12 and  $S_2 := C_{f,e} * C_2 * C_{e,f} = M = C_{e,e}$ . Notice that  $\mathbb{I}_f = (\frac{1}{2}, 1]$ . Then for any  $t \in (0, \frac{1}{2})$  which is full in  $\mathbb{I} \setminus \mathbb{I}_f$ ,  $s_{\frac{1}{2}}^{-1}(t) = t + \frac{1}{2} \in \mathbb{I}_f$  but  $e^{-1}(t) = t \notin \mathbb{I}_f$ . So by Theorem 15,  $C_1$  is factorizable but  $C_2$  is not. In fact, by a direct calculation  $C_2 = C_{e,f} *_{\text{W}} C_{f,e}$ .

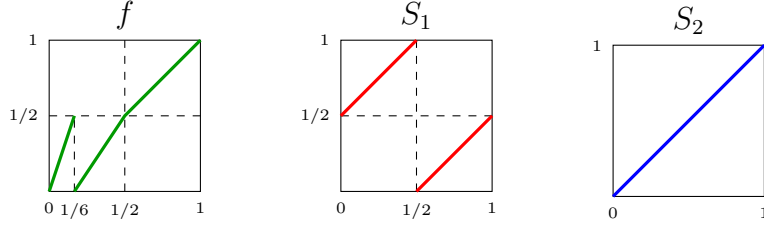


Figure 4: The graph of  $f$  corresponding to the associated  $\sigma$ -algebras of  $C_1$  and  $C_2$ , as well as supports of  $S_1$  and  $S_2$ , respectively.

From the proof of Lemma 8, for  $\alpha, \beta \in \mathcal{T}$  satisfying  $\lambda(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) = 1$ , every  $\mathcal{A}$ -product of  $C_{e,\alpha}$  and  $C_{\beta,e}$  equals  $C_{e,\alpha} * C_{\beta,e}$ , which is extreme. So the joining copulas do not affect the extremality of implicit dependence copulas for  $\alpha, \beta$  whose images of the injective parts cover  $\mathbb{I}$  a.e. However, when  $\lambda(\mathbb{I}_\alpha \cup \mathbb{I}_\beta) < 1$ , the factorizable copula  $C_{e,\alpha} * C_{\beta,e}$  is not extreme from Theorem 5. Together with Lemma 13, we wonder whether there exists an extreme copula  $C$  in  $\mathcal{C}_{\text{ID}}$  such that  $\mu_C \ll \mu_{C_{e,\alpha} * C_{\beta,e}}$ . The answer is affirmative by another interesting consequence of Theorem 6 that any copula of the form  $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$  where  $\mathcal{A} = \{M\}$  or  $\{W\}$  is extreme. This result follows from the fact that  $M$  and  $W$  are pointwise upper and lower bounds, respectively, among bivariate copulas.

**Theorem 17.** *Let  $\alpha, \beta \in \mathcal{T}$ . Then  $C_{e,\alpha} *_M C_{\beta,e}$  and  $C_{e,\alpha} *_W C_{\beta,e}$  are extreme.*

*Proof.* We prove only that  $C := C_{e,\alpha} *_M C_{\beta,e}$  is extreme since the other can be shown similarly. Let  $C_1, C_2 \in \mathcal{C}$  be such that  $C = kC_1 + (1-k)C_2$  for some  $k \in (0, 1)$ . Then for each  $i \in \{1, 2\}$ ,  $\mu_{C_i} \ll \mu_C$  which implies, by the first part of Theorem 6, that  $C_i$  is the copula of implicitly dependent random variables  $X_i, Y_i \sim \mathcal{U}(0, 1)$  such that  $\alpha(X_i) = \beta(Y_i)$  a.s. Moreover, the second part of the same theorem guarantees the existence of  $\mathcal{A} = \{A_t\}_{t \in \mathbb{I}}$  and  $\mathcal{B} = \{B_t\}_{t \in \mathbb{I}}$  satisfying  $C_1 = C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$  and  $C_2 = C_{e,\alpha} *_{\mathcal{B}} C_{\beta,e}$ . Now, if  $C_1(x, y) \neq C(x, y)$  for some  $(x, y) \in \mathbb{I}^2$ , then by the definition of generalized Markov product, inequalities  $A_t \leq M$  and  $B_t \leq M$ , for all  $t \in \mathbb{I}$ , imply that  $C_1(x, y) < C(x, y)$  and  $C_2(x, y) \leq C(x, y)$ , respectively. This gives the expression  $kC_1(x, y) + (1-k)C_2(x, y) < C(x, y)$ , which is a contradiction. Thus,  $C_1 = C$ . Observe that  $C_2 = C$  can be shown in a similar manner, so  $C = C_{e,\alpha} *_M C_{\beta,e}$  is extreme as desired.  $\square$

Next, some subclasses of implicit dependence copulas are given in Example 18-20. The first two demonstrate consequences of Theorem 6 by writing a symmetric hairpin copula  $C_f$  and a line-in-square copula  $C_r$ , both of which are extreme implicit dependence copulas (see [2, 23]), as generalized Markov product of complete dependence copulas:

$$C_f = C_{e,\alpha} *_M C_{\beta,e} \quad \text{and} \quad C_r = C_{e,\alpha} *_{C_r} C_{\beta,e},$$

where  $\alpha, \beta \in \mathcal{T}$  in each example are obtained from associated  $\sigma$ -algebras of the implicit dependence copula. Notice that the extremality of  $C_f$  can also be obtained by Theorem 17 while the corresponding joining copulas of  $C_r$  can be chosen to be itself, but not  $M$  or  $W$ . On the other hand, the last example shows that for any copula  $C \neq M$  or  $W$ , if  $C(a, b) = kM(a, b) + (1-k)W(a, b)$  for some  $(a, b) \in (0, 1)^2$  and  $k \in (0, 1)$  then the  $C$ -product of two simple complete dependence copulas corresponding to  $\Lambda_a, \Lambda_b \in \mathcal{T}$ , where  $\Lambda_\theta(x) := \min\{\frac{x}{\theta}, \frac{1-x}{1-\theta}\}$ , is not extreme. Denoting  $D_C^{(a,b)} := C_{e,\Lambda_a} *_C C_{\Lambda_b,e}$ , we shall verify that

$$D_C^{(a,b)} = kD_M^{(a,b)} + (1-k)D_W^{(a,b)}. \quad (4.1)$$

Hence,  $A = M$  and  $W$  are only two copulas that guarantee the extremality of  $C_{e,\alpha} *_A C_{\beta,e}$  for all  $\alpha, \beta \in \mathcal{T}$ . Details of these examples are described below. From now on, we denote  $\text{Gr}(f) := \{(x, f(x)) : x \in \text{dom}(f)\}$  and  $\widetilde{\text{Gr}}(g) := \{(g(y), y) : y \in \text{dom}(g)\}$ .

**Example 18.** Let  $f: \mathbb{I} \rightarrow \mathbb{I}$  be a homeomorphism such that  $f(x) < x$  for all  $x \in (0, 1)$  and  $C_f$  denote the symmetric hairpin copula with  $\text{supp}(C_f) = \text{Gr}(f) \cup \widetilde{\text{Gr}}(f)$ . Then  $\delta: x \mapsto C_f(x, x)$ , the so-called *diagonal section* of  $C_f$ , gives another form of  $C_f$  as  $C_f(x, y) = \min\{x, y, \frac{1}{2}(\delta(x) + \delta(y))\}$  for all  $(x, y) \in \mathbb{I}^2$ , see Nelsen and Fredricks [16].

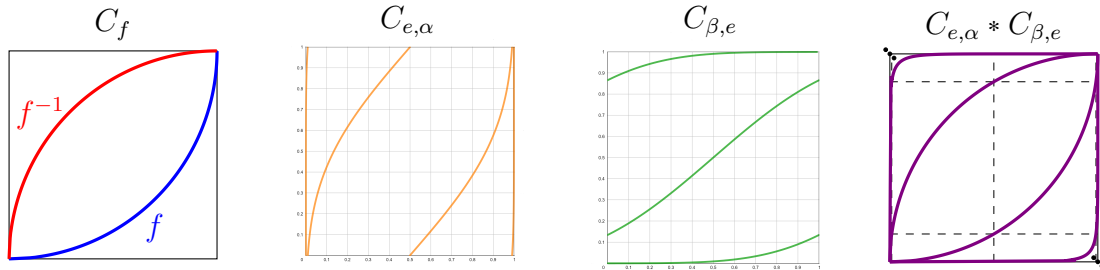


Figure 5: Supports of  $C_f, C_{e,\alpha}, C_{\beta,e}$  and  $C_{e,\alpha} * C_{\beta,e}$  respectively where  $f(x) = 1 - \sqrt{1 - x^2}$ .

Measure-preserving transformations  $\alpha$  and  $\beta$  are derived in [19].

$$\begin{aligned} \alpha(x) &= \gamma(f^{-2n}(x)), & \text{for } x \in I_n &:= \left[ f^{2n+2}\left(\frac{1}{2}\right), f^{2n}\left(\frac{1}{2}\right) \right] \quad \text{and} \\ \beta(y) &= \gamma(f^{-2n+1}(y)), & \text{for } y \in J_n &:= \left[ f^{2n+1}\left(\frac{1}{2}\right), f^{2n-1}\left(\frac{1}{2}\right) \right], \end{aligned}$$

where  $n \in \mathbb{Z}$  and  $\gamma(t) := \lambda(\bigcup_{n \in \mathbb{Z}} f^{2n}[f^2(\frac{1}{2}), t])$  for  $t \in I_0$ . For convenience, let us denote  $\alpha_n := \alpha|_{I_n}$  and  $\beta_n := \beta|_{J_n}$ . Recall that  $f^k := f \circ f^{k-1}$ ,  $f^{-k} := (f^{-1})^k$  and  $f^0 :=$  the identity map for  $k \in \mathbb{N}$ . By [18, Example 12] and applications of the chain rule, we have the following results for  $s \in I_0$ .

- Denote  $r := f^{-1}(s)$  so that  $\alpha(s) = \beta(r)$ . Then for any  $k \in \mathbb{Z}$ , we can define  $s_k := \alpha_k^{-1} \circ \alpha(s)$  and  $r_k := \beta_k^{-1} \circ \beta(r)$ . Moreover,

$$\alpha'(s_k) = \frac{d}{ds_k} (\alpha \circ f^{-2k})(s_k) = \begin{cases} \gamma'(s) \prod_{i=0}^{2k-1} \frac{1}{f'(f^i(s))} & \text{if } k > 0; \\ \gamma'(s) & \text{if } k = 0; \\ \gamma'(s) \prod_{i=1}^{-2k} f'(f^{-i}(s)) & \text{if } k < 0. \end{cases}$$

- For any  $x \in I_k$ ,  $\partial_2 C_{e,\alpha}(x, \alpha(s)) = \mu_k^* \mathbb{1}_{[0, s_k)}(x) + \mu_k \mathbb{1}_{(s_k, 1]}(x)$  where  $\mu_k^* = \mu_{k+1}$  and

$$\mu_k := \begin{cases} \frac{1}{\gamma'(s)} \sum_{i=k}^{\infty} \prod_{p=0}^{2i-1} f'(f^p(s)) & \text{if } k > 0; \\ \frac{1}{\gamma'(s)} \left[ \sum_{i=k}^0 \prod_{p=1}^{-2i} \frac{1}{f'(f^{-p}(s))} + \sum_{i=1}^{\infty} \prod_{p=0}^{2i-1} f'(f^p(s)) \right] & \text{if } k \leq 0. \end{cases}$$

- For any  $y \in J_\ell$ ,  $\partial_2 C_{e,\beta}(y, \alpha(s)) = \eta_\ell^* \mathbb{1}_{[0, r_\ell)}(y) + \eta_\ell \mathbb{1}_{(r_\ell, 1]}(y)$  where  $\eta_\ell^* = \eta_{\ell+1}$  and

$$\eta_\ell := \begin{cases} \frac{1}{\gamma'(s)} \sum_{j=\ell}^{\infty} \prod_{q=0}^{2j-2} f'(f^q(s)) & \text{if } \ell > 0; \\ \frac{1}{\gamma'(s)} \left[ \sum_{j=\ell}^0 \prod_{q=1}^{-2j+1} \frac{1}{f'(f^{-q}(s))} + \sum_{j=1}^{\infty} \prod_{q=0}^{2j-2} f'(f^q(s)) \right] & \text{if } \ell \leq 0. \end{cases}$$

$$\bullet A_{\alpha(s)}(\mu_k, \eta_\ell) = \begin{cases} \mu_\ell + \frac{\delta'(s_{\ell-1})}{2\alpha'(s_{\ell-1})} & \text{if } k < \ell; \\ \mu_k & \text{if } k \geq \ell. \end{cases}$$

By tedious but straightforward calculations,

$$(\eta_\ell - \mu_\ell) \alpha'(s_{\ell-1}) = \sum_{i=1}^{\infty} (-1)^{i+1} (f^i)'(s_{\ell-1}) \text{ for any } \ell \in \mathbb{Z}. \quad (4.2)$$

The easiest case is when  $\ell = 1$ , which gives  $\alpha'(s) = \gamma'(s)$  and, by the chain rule,

$$\begin{aligned} \eta_1 - \mu_1 &= \frac{1}{\gamma'(s)} \sum_{i=1}^{\infty} \left[ \prod_{p=0}^{2i-2} f'(f^p(s)) - \prod_{p=0}^{2i-1} f'(f^p(s)) \right] \\ &= \frac{1}{\gamma'(s)} \sum_{i=1}^{\infty} \left[ (f^{2i-1})'(s) - (f^{2i})'(s) \right] = \frac{1}{\gamma'(s)} \sum_{i=1}^{\infty} (-1)^{i+1} (f^i)'(s). \end{aligned}$$

So (4.2) holds for  $\ell = 1$ . By Chamizo et al. [5, Theorem 4.1], the expression (4.2) implies that  $(\eta_\ell - \mu_\ell) \alpha'(s_{\ell-1}) = \frac{\delta'(s_{\ell-1})}{2}$ . Hence,  $\mu_\ell + \frac{\delta'(s_{\ell-1})}{2\alpha'(s_{\ell-1})} = \eta_\ell$  and  $A_{\alpha(s)}(\mu_k, \eta_\ell) = \eta_\ell \mathbb{1}_{k < \ell} + \mu_k \mathbb{1}_{k \geq \ell}$ . Finally, by the increasing and Lipschitz properties of  $\delta$ , we have  $\mu_\ell \leq \eta_\ell \leq \mu_{\ell-1}$  for  $\ell \in \mathbb{Z}$  which also implies that we can choose  $A_{\alpha(s)} = M$  for all  $s \in I_0$ . Therefore,  $C_f = C_{e,\alpha} * M C_{\beta,e}$ .

**Example 19.** Let  $C_r$ , where  $\frac{1}{2} < r < 1$ , be a copula, called a line-in-square copula whose support is shown in Figure 6.

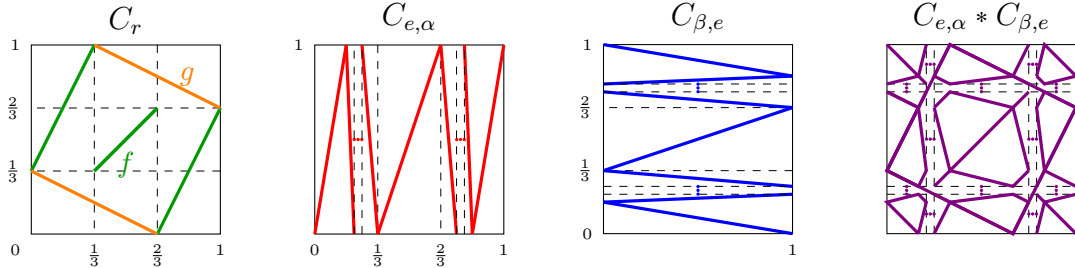


Figure 6: Supports of  $C_r$ ,  $C_{e,\alpha}$ ,  $C_{\beta,e}$  and  $C_{e,\alpha} * C_{\beta,e}$  respectively where  $r = \frac{2}{3}$

Denote  $a := \frac{r}{1-r}$ . Following [13],  $\text{supp}(C_r) = \text{Gr}(f) \dot{\cup} \widetilde{\text{Gr}}(g)$  where

$$f(x) = \begin{cases} ax + 1 - r & \text{if } x \in [0, 1 - r]; \\ x & \text{if } x \in (1 - r, r); \\ ax - ar & \text{if } x \in [r, 1] \end{cases} \quad \text{and} \quad g(y) = \begin{cases} r - ay & \text{if } y \in [0, 1 - r]; \\ 1 + ar - ay & \text{if } y \in [r, 1]. \end{cases} \quad (4.3)$$

The functions  $\alpha, \beta \in \mathcal{T}$  and their corresponding partitions can be defined using the concept of *limb numbering system* [13]. We define  $\alpha$  and  $\beta$  on sets  $I_0, I_{\pm 1}, I_{\pm 2}, \dots, I_{\pm k}, \dots$  and  $J_0, J_{\pm 1}, J_{\pm 2}, \dots, J_{\pm k}, \dots$  inductively on  $k$ . For convenience, let us denote  $\alpha_m := \alpha|_{I_m}$  and  $\beta_m := \beta|_{J_m}$ . First, let  $\alpha_0(x) := \frac{x+r-1}{2r-1}$ ,  $\beta_0(y) := \frac{y+r-1}{2r-1}$  and  $I_0 = J_0 := (1 - r, r)$ . For each  $k \in \mathbb{N}$ , if  $\alpha_{\pm(k-1)}, \beta_{\pm(k-1)}, I_{\pm(k-1)}$  and  $J_{\pm(k-1)}$  are defined, let

$$\begin{aligned} \alpha_{-k}(x) &= \beta_{k-1} \circ (f|_{(0,1-r)})(x) & \text{with } I_{-k} &= \alpha_{-k}^{-1}((0,1)); \\ \alpha_k(x) &= \beta_{-(k-1)} \circ (f|_{(r,1)})(x) & \text{with } I_k &= \alpha_k^{-1}((0,1)); \end{aligned} \quad (4.4)$$

$$\begin{aligned}\beta_{-k}(y) &= \alpha_{-(k-1)} \circ (g|_{(0,1-r)})(y) & \text{with } J_{-k} &= \beta_{-k}^{-1}((0,1)); \\ \beta_k(y) &= \alpha_{k-1} \circ (g|_{(r,1)})(y) & \text{with } J_k &= \beta_k^{-1}((0,1)).\end{aligned}$$

The supports of  $C_{e,\alpha}$  and  $C_{\beta,e}$  for  $C_r$  when  $r = \frac{2}{3}$  are shown in Figure 6. By induction on  $|k|$ ,  $\alpha_k$  and  $\beta_k$  are linear functions from  $I_k = J_k$  onto  $(0,1)$  whose slopes are  $\pm \frac{a^{|k|}}{2r-1}$ . This is because  $f$  and  $g$  are linear with slopes  $a$  and  $-a$ , respectively, on  $(0,1-r)$  and  $(r,1)$ . Moreover, by the computations in Section 6, the following hold for  $s \in I_0$ .

A. By [18, Remark 4], for  $m \in \mathbb{Z}$ ,  $x \in I_m$  and  $y \in J_m$ ,  $\partial_2 C_{e,\alpha}(x, \alpha(s)) = \mu_m^* \mathbb{1}_{[0,s_m]}(x) + \mu_m \mathbb{1}_{(s_m,1]}(x)$  and  $\partial_2 C_{e,\beta}(y, \alpha(s)) = \eta_m^* \mathbb{1}_{[0,t_m]}(y) + \eta_m \mathbb{1}_{(t_m,1]}(y)$  where  $s_m := \alpha_m^{-1} \circ \alpha(s)$ ,  $t_m := \beta_m^{-1} \circ \beta(s)$ ,  $\mu_m$  and  $\eta_m$  are both equal to

$$\begin{cases} ac_k & \text{if } m = -(2k-1) < 0; \\ 1-r-c_{k-1} & \text{if } m = -2k < 0; \\ r+c_k & \text{if } m = 2k \geq 0; \\ 1-ac_{k-1} & \text{if } m = 2k-1 > 0, \end{cases} \quad (4.5)$$

where  $c_k = \sum_{i=1}^k \frac{2r-1}{a^{2i}}$  for  $k \in \mathbb{N} \cup \{0\}$ , and  $\mu_m^* = \eta_m^* = \mu_{m-2} \mathbb{1}_{2\mathbb{Z}}(m) + \mu_{m+2} \mathbb{1}_{\mathbb{Z} \setminus \{2\mathbb{Z} \cup \{-1\}\}}(m)$ .

B. If  $\ell \geq 0$  then

$$A_{\alpha(s)}(\mu_k, \eta_\ell) = \begin{cases} \min \{ \mu_k, (1-r)(\mu_k + \eta_\ell - (1-r)) \} & \text{if } k < 0; \\ \max \{ r\mu_k + (1-r)\eta_\ell - r(1-r), \mu_k + \eta_\ell - 1 \} & \text{if } k \geq 0. \end{cases} \quad (4.6)$$

If  $\ell < 0$  then

$$A_{\alpha(s)}(\mu_k, \eta_\ell) = \begin{cases} \max \{ (1-r)\mu_k + r\eta_\ell - r(1-r), 0 \} & \text{if } k < 0; \\ \min \{ r(\mu_k + \eta_\ell - r), \eta_\ell \} & \text{if } k \geq 0. \end{cases} \quad (4.7)$$

It is straightforward to check that  $A_{\alpha(s)}(\mu_k, \eta_\ell) = C_r(\mu_k, \eta_\ell)$  for all  $k, \ell$ . So we can choose  $A_{\alpha(s)} = C_r$  for all  $s \in I_0$  and it yields that  $C_r = C_{e,\alpha} *_{C_r} C_{\beta,e}$ .

**Example 20.** Let  $C \in \mathcal{C} \setminus \{M, W\}$  be such that  $C(a, b) = kM(a, b) + (1-k)W(a, b)$  for some  $(a, b) \in (0,1)^2$  and  $k \in (0,1)$ . Define  $\Lambda_\theta: x \mapsto \frac{x}{\theta} \mathbb{1}_{[0,\theta]}(x) + \frac{1-x}{1-\theta} \mathbb{1}_{(\theta,1]}(x)$  for  $\theta \in (0,1)$ . To determine  $D_C^{(a,b)} := C_{e,\Lambda_a} *_{C} C_{\Lambda_b,e}$ , let us recall the following.

- $\forall z \in \mathbb{I}$ ,  $\partial_2 C_{e,\Lambda_a}(z, t) \in \{0, a, 1\}$  and  $\partial_1 C_{\Lambda_b,e}(t, z) \in \{0, b, 1\}$  for a.e.  $t \in \mathbb{I}$ .
- The integrand in  $D_C^{(a,b)}$ , namely  $C(\partial_2 C_{e,\Lambda_a}(x, t), \partial_1 C_{\Lambda_b,e}(t, y))$ , becomes a simple function of finite range.

By straightforward calculations, the formula for  $D_C^{(a,b)}(x, y)$  depends on which  $I_i \times J_j(x, y)$  belongs to, where  $I_1 = [0, a]$ ,  $I_2 = (a, 1]$ ,  $J_1 = [0, b]$  and  $J_2 = (b, 1]$ . Each  $I_i \times J_j$  is further divided into two regions by a diagonal line defined by  $\Lambda_a(x) = \Lambda_b(y)$ . For instance, with  $(x, y) \in I_2 \times J_2$ , if  $\frac{1-x}{1-a} = \Lambda_a(x) \leq \Lambda_b(y) = \frac{1-y}{1-b}$ , then

$$D_C^{(a,b)}(x, y) = \int_0^{\Lambda_a(x)} C(a, b) dt + \int_{\Lambda_a(x)}^{\Lambda_b(y)} C(1, b) dt + \int_{\Lambda_b(y)}^1 C(1, 1) dt = y - \Lambda_a(x)(b - C(a, b)).$$

On the other hand, if  $\Lambda_a(x) > \Lambda_b(y)$ , then

$$D_C^{(a,b)}(x, y) = \int_0^{\Lambda_b(y)} C(a, b) dt + \int_{\Lambda_b(y)}^{\Lambda_a(x)} C(a, 1) dt + \int_{\Lambda_a(x)}^1 C(1, 1) dt = x - \Lambda_b(y)(a - C(a, b)).$$

Since  $C(a, b) = kM(a, b) + (1-k)W(a, b)$ , we can write  $D_C^{(a,b)} = kD_M^{(a,b)} + (1-k)D_W^{(a,b)}$  which is not extreme regardless of the extremality of  $C$ .

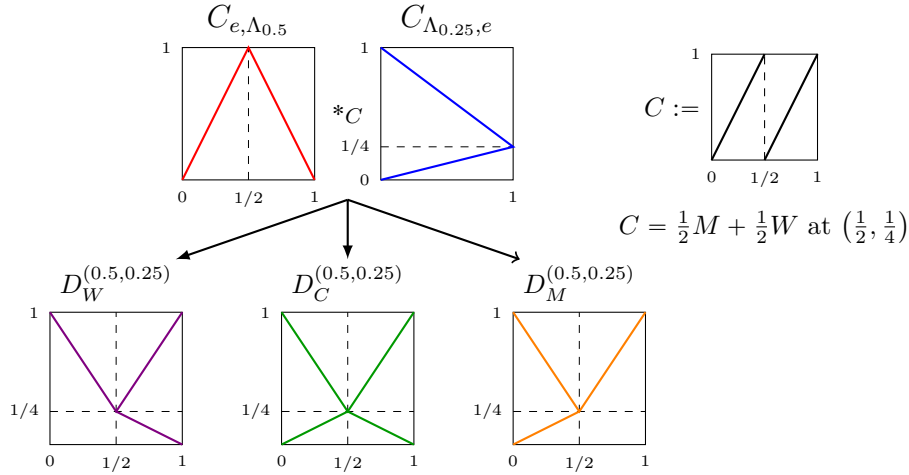


Figure 7: Supports of  $C_{e, \Lambda_{0.5}}, C_{\Lambda_{0.25, e}}$  and their  $*_C$ -product, respectively, in Example 20

## 5 Conclusion

This manuscript is a continuation of the work in [24, 20, 19]. Building on ideas in [19] and borrowing tools from [20, 18], we prove that a factorizable copula  $C_{e, \alpha} * C_{\beta, e}$  is extreme if and only if the images of the injective parts of  $\alpha$  and  $\beta$  cover  $[0, 1]$  a.e. (Theorem 9). The factorizability of extreme non-atomic copulas is also characterized in Theorem 15. The extremality of implicit dependence copulas is much more challenging to characterize. However, we show that every implicit dependence copulas whose joining copulas are either all  $M$ 's or all  $W$ 's is always extreme (Theorem 17). As examples of implicit dependence copulas, we investigate a symmetric hairpin copula [19] and a line-in-square copula  $C_r$  [2]. Tedious calculations show that their joining copulas are  $M$  and  $C_r$  itself, respectively. Figure 8 contains a diagram illustrating relationships among types of copulas referred to in this manuscript.

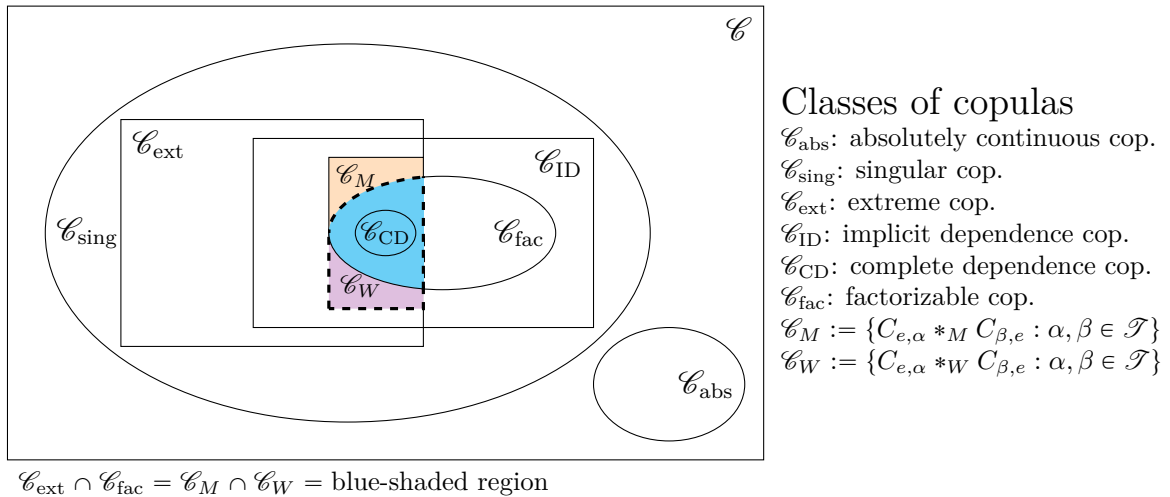


Figure 8: Relationships among types of copulas in this manuscript

## 6 Appendix: Detailed calculations in Example 19

For two intervals  $K_1$  and  $K_2$  in a partition of  $[0, 1]$ ,  $K_1 < K_2$  if  $K_1$  is on the left-hand side of  $K_2$  and  $K_1 \leq K_2$  if  $K_1 < K_2$  or  $K_1 = K_2$ . Denote

$$a := \frac{r}{1-r}, \quad c_0 := 0, \quad \text{and} \quad c_k := (2r-1) \sum_{i=1}^k \frac{1}{a^{2i}} = (1-r)^2 \left( 1 - \frac{(1-r)^{2k}}{r^{2k}} \right)$$

for  $k \in \mathbb{N}$ . Note that  $\lim_{k \rightarrow \infty} c_k = (1-r)^2$ .

By induction on  $k \in \mathbb{N}$ , repeated applications of (4.4) yield

$$I_{-(2k-1)} = J_{-(2k-1)} = (ac_{k-1}, ac_k), \quad I_{-2k} = J_{-2k} = (1-r-c_k, 1-r-c_{k-1}) \quad (6.1)$$

$$\text{and } I_{2k-1} = J_{2k-1} = (1-ac_k, 1-ac_{k-1}), \quad I_{2k} = J_{2k} = (r+c_{k-1}, r+c_k). \quad (6.2)$$

For  $k=1$ ,  $\alpha_{-1}(x) = \beta_0(ax+1-r) = \frac{x}{ac_1}$  and  $\beta_{-1}(y) = \alpha_0(r-ay) = 1 - \frac{y}{ac_1}$  which give  $I_{-1} = J_{-1} = (0, ac_1)$ ;  $\alpha_1(x) = 1 - \frac{1-x}{ac_1}$  and  $\beta_1(y) = \frac{1-y}{ac_1}$  which give  $I_1 = J_1 = (1-ac_1, 1)$ ;  $\alpha_{-2}(x) = \beta_{-2}(x) = \frac{1-r-x}{c_1}$  which yields  $I_{-2} = J_{-2} = (1-r-c_1, 1-r)$ ; and  $\alpha_2(x) = \beta_2(x) = 1 - \frac{x-r}{c_1}$  which yields  $I_2 = J_2 = (r, r+c_1)$ . Suppose that (6.1)-(6.2) hold for a fixed  $k \in \mathbb{N}$ . By (4.4),

$$\begin{aligned} x \in I_{-(2k+1)} &\Leftrightarrow ax+1-r \in J_{2k} \Leftrightarrow x \in (ac_k, ac_{k+1}); \\ y \in J_{-(2k+1)} &\Leftrightarrow r-ay \in I_{-2k} \Leftrightarrow y \in (ac_k, ac_{k+1}); \\ x \in I_{2k+1} &\Leftrightarrow a(x-r) \in J_{-2k} \Leftrightarrow x \in (1-ac_{k+1}, 1-ac_k); \\ y \in J_{2k+1} &\Leftrightarrow 1+ar-ay \in I_{2k} \Leftrightarrow y \in (1-ac_{k+1}, 1-ac_k), \end{aligned}$$

where we have used the identity  $(2r-1) + c_{k-1} = a^2 c_k$ . Similarly,  $I_{-(2k+2)}, J_{-(2k+2)}, I_{2k+2}$  and  $J_{2k+2}$  can be shown to satisfy (6.1) and (6.2).

As a consequence of (6.1) and (6.2),  $\lambda(I_{\pm 2k}) = \Delta c_k := c_k - c_{k-1}$  and  $\lambda(I_{\pm(2k-1)}) = a\Delta c_k$ . Also,  $\bigcup_{k=1}^{\infty} I_{-(2k-1)} = (0, r(1-r))$ ,  $\bigcup_{k=1}^{\infty} I_{-2k} = (r(1-r), 1-r)$ ,  $\bigcup_{k=1}^{\infty} I_{2k} = (r, 1-r(1-r))$  and  $\bigcup_{k=1}^{\infty} I_{2k-1} = (1-r(1-r), 1)$  except for the countably many endpoints of the  $I_m$ 's. Moreover, if  $m, q$  are odd and  $n, p$  are even, then

$$I_{-m} < I_{-(m+2)} < I_{-(n+2)} < I_{-n} < I_0 < I_p < I_{p+2} < I_{q+2} < I_q. \quad (6.3)$$

Let us denote  $f_1, f_2, g_1$  and  $g_2$  so that  $f = f_1 \mathbb{1}_{(0,1-r)} + f_2 \mathbb{1}_{(r,1)}$  and  $g = g_1 \mathbb{1}_{(0,1-r)} + g_2 \mathbb{1}_{(r,1)}$ . Observe that

$$\begin{aligned} f_1(I_{-(2k+1)}) = I_{2k} &= g_2(I_{2k+1}), \quad f_1(I_{-2k}) = I_{2k-1} = g_2(I_{2k}), \\ f_2(I_{2k+1}) = I_{-2k} &= g_1(I_{-(2k+1)}), \quad f_2(I_{2k}) = I_{-(2k-1)} = g_1(I_{-2k}). \end{aligned} \quad (6.4)$$

To verify (4.5), let  $s \in I_0$  and, for  $m \in \mathbb{Z}$ , define  $s_m := \alpha_m^{-1} \circ \alpha(s)$  and  $t_m := \beta_m^{-1} \circ \beta(s)$ . By [18, Remark 4],  $\mu_m := \sum_{I_i \leq I_m} \frac{1}{|\alpha'(s_i)|} = \sum_{I_i \leq I_m} \lambda(I_i)$  and  $\eta_m := \sum_{J_i \leq J_m} \frac{1}{|\beta'(t_i)|} = \sum_{J_i \leq J_m} \lambda(J_i) = \mu_m$ , which are independent from  $s$ .

- **Case 1**  $m = -(2k-1) < 0$ : From (6.1),  $\mu_m = \sum_{i=1}^k a\Delta c_i = ac_k$ .
- **Case 2**  $m = -2k \leq 0$ : If  $k > 0$  then by (6.1), we have

$$\mu_m = \sum_{i=1}^{\infty} a\Delta c_i + \sum_{i=k}^{\infty} \Delta c_i = a(1-r)^2 + (1-r)^2 - c_{k-1} = 1-r-c_{k-1}.$$

If  $k=0$ , then  $\mu_0 = \mu_{-2} + \lambda(I_0) = 1-r-c_0 + (2r-1) = r$ .

- **Case 3**  $m = 2k > 0$ : Since  $\mu_0 = r$ , (6.2) yields  $\mu_m = \mu_0 + \sum_{i=1}^k \Delta c_i = r + c_k$ .
- **Case 4**  $m = 2k - 1 > 0$ : By (6.2),

$$\mu_m = \mu_0 + \sum_{i=1}^{\infty} \Delta c_i + \sum_{i=k}^{\infty} a \Delta c_i = r + (1-r)^2 + a((1-r)^2 - c_{k-1}) = 1 - ac_{k-1}.$$

Defining  $\mu_m^* := \sum_{I_i < I_m} \frac{1}{|\alpha'(s_i)|}$  and  $\eta_m^* := \sum_{J_i < J_m} \frac{1}{|\beta'(t_i)|}$ , it is easy to derive from (6.3) that

$$\mu_m^* = \eta_m^* = \begin{cases} 0 & \text{if } m = -1; \\ \mu_{m-2} & \text{if } m \text{ is even;} \\ \mu_{m+2} & \text{otherwise,} \end{cases} \quad (6.5)$$

and hence for every  $m \in \mathbb{Z}$ ,  $I_m$  and  $J_m$  can be written as  $I_m = (\mu_m^*, \mu_m)$  and  $J_m = (\eta_m^*, \eta_m)$ , respectively. Next, we use (4.3) to obtain

$$\begin{aligned} \text{for } x \in [0, 1-r] \cup [r, 1], & \quad f(1-x) = 1 - f(x) = g(x) \quad \text{and} \\ \text{for } x \in [0, r], & \quad g_2^{-1}(1-x) = 1 - g_1^{-1}(x). \end{aligned} \quad (6.6)$$

In particular, using (4.5) and (6.5), we can consider by cases to show that for  $m \in \mathbb{Z}$ ,

$$\mu_m^* = 1 - \mu_{-m}, \quad f(\mu_m) = \begin{cases} \eta_{-m-1} & \text{if } m < 0; \\ 0 & \text{if } m = 0; \\ \eta_{1-m} & \text{if } m > 0, \end{cases} \quad \text{and} \quad g(\eta_m) = \begin{cases} \mu_{m+1}^* & \text{if } m < 0; \\ 1 & \text{if } m = 0; \\ \mu_{m-1}^* & \text{if } m > 0. \end{cases} \quad (6.7)$$

Now, for each  $s \in I_0$  and  $m, \ell \in \mathbb{Z}$ , the formula  $A_{\alpha(s)}(\mu_m, \eta_\ell)$  in [18, Lemma 8] can be expressed as

$$A_{\alpha(s)}(\mu_m, \eta_\ell) = \sum_{I_i \leq I_m} \frac{1}{|\alpha'_i(s_i)|} T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(s_i) = \sum_{I_i \leq I_m} (\mu_i - \mu_i^*) T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(s_i), \quad (6.8)$$

where for any  $h \in L^1(\mathbb{I})$ ,

$$T_{C_r} h(x) = \begin{cases} (1-r)h(g_1^{-1}(x)) + rh(f_1(x)) & \text{if } 0 \leq x < 1-r; \\ (1-r)h(g_1^{-1}(x)) + (2r-1)h(x) + (1-r)h(g_2^{-1}(x)) & \text{if } 1-r \leq x < r; \\ rh(f_2(x)) + (1-r)h(g_2^{-1}(x)) & \text{if } r \leq x \leq 1. \end{cases} \quad (6.9)$$

In particular, when  $h = \mathbb{1}_{[0, \eta_\ell]}$ ,  $\ell \in \mathbb{Z}$ , we use (6.6) to show that for  $i \in \mathbb{Z} \setminus \{0\}$ ,

$$\begin{aligned} 1 - T_{C_r} \mathbb{1}_{[0, 1-\eta_\ell]}(s_i) &= (1-r)(1 - \mathbb{1}_{[0, 1-\eta_\ell]}(g^{-1}(s_i))) + r(1 - \mathbb{1}_{[0, 1-\eta_\ell]}(f(s_i))) \\ &= (1-r)\mathbb{1}_{(1-\eta_\ell, 1]}(g^{-1}(s_i)) + r\mathbb{1}_{(1-\eta_\ell, 1]}(f(s_i)) \\ &= (1-r)\mathbb{1}_{[0, \eta_\ell]}(g^{-1}(1-s_i)) + r\mathbb{1}_{[0, \eta_\ell]}(f(1-s_i)) \\ &= T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(1-s_i) = T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(1-s_i), \end{aligned} \quad (6.10)$$

and similarly,  $1 - T_{C_r} \mathbb{1}_{[0, 1-\eta_\ell]}(s) = T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(1-s)$ . This implies

$$\mu_m + \eta_\ell - 1 + A_{\alpha(s)}(\mu_{-m}^*, \eta_{-\ell}^*) = \eta_\ell - A_{\alpha(s)}(\mu_{-m}^*, 1) + A_{\alpha(s)}(\mu_{-m}^*, \eta_{-\ell}^*)$$

$$\begin{aligned}
&= \eta_\ell - \sum_{I_i < I_{-m}} \lambda(I_i) (1 - T_{C_r} \mathbb{1}_{[0, 1-\eta_\ell]}(s_i)) && \text{(by (6.8))} \\
&= \eta_\ell - \sum_{I_i < I_{-m}} \lambda(I_i) T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(1 - s_i) && \text{(by (6.10))} \\
&= \eta_\ell - \sum_{I_j > I_m} \lambda(I_j) T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(u_j) \quad (j := -i, u_j := 1 - s_i) \\
&= A_{\alpha(s)}(1, \eta_\ell) - \sum_{I_j > I_m} \lambda(I_j) T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(s_j) \quad \text{(by (6.9), (6.4))} \\
&= \sum_{I_j \leq I_m} \lambda(I_j) T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(s_j) = A_{\alpha(s)}(\mu_m, \eta_\ell). && \text{(6.11)}
\end{aligned}$$

To verify  $A_{\alpha(s)}(\mu_m, \eta_\ell)$  in (4.6)-(4.7), we start from the case  $\ell < 0$  which gives  $f(\mu_{1-\ell}) = \eta_\ell$  and  $g(\eta_\ell) = \mu_{\ell+1}^*$ . Note that  $\mu_{\ell+1}^* \leq 1 - r < r < \mu_{1-\ell}$ , so (6.9), (6.4) and the fact that  $s_i \in I_i$  imply

$$T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(s_i) = \begin{cases} 1 - r & \text{if } I_{\ell+1} \leq I_i \leq I_0; \\ r & \text{if } I_0 < I_i \leq I_{1-\ell}; \\ 0 & \text{otherwise.} \end{cases} \quad (6.12)$$

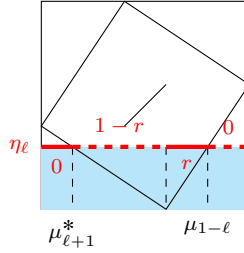


Figure 9:  $T_{C_r} \mathbb{1}_{[0, \eta_\ell]}(s)$  in the case  $\ell < 0$

If  $m \leq 0$ , we apply  $\mu_{-\ell-1} = f(\mu_\ell) = a\mu_\ell + 1 - r$  (see (6.7)) to obtain

$$\begin{aligned}
A_{\alpha(s)}(\mu_m, \mu_\ell) &= \max \left\{ (1-r)(\mu_m - \mu_{\ell+1}^*), 0 \right\} = \max \left\{ (1-r)(\mu_m + \mu_{-\ell-1} - 1), 0 \right\} \\
&= \max \left\{ (1-r)(\mu_m + a\mu_\ell - r), 0 \right\} = \max \left\{ (1-r)\mu_m + r\mu_\ell - r(1-r), 0 \right\}.
\end{aligned}$$

In particular,  $A_{\alpha(s)}(\mu_0, \mu_\ell) = (1-r)r + r\mu_\ell - r(1-r) = r\mu_\ell$ . On the other hand, if  $m > 0$ , then by (6.7),  $\mu_\ell = f(\mu_{1-\ell}) = a(\mu_{1-\ell} - r)$ , i.e.,  $\mu_{1-\ell} = \frac{\mu_\ell}{a} + r$ , so

$$\begin{aligned}
A_{\alpha(s)}(\mu_m, \mu_\ell) &= A_{\alpha(s)}(\mu_0, \mu_\ell) + r(\min \{ \mu_m, \mu_{1-\ell} \} - \mu_0) = r(\min \{ \mu_m, \mu_{1-\ell} \} + \mu_\ell - r) \\
&= \min \left\{ r(\mu_m + \mu_\ell - r), r \left( \frac{\mu_\ell}{a} + \mu_\ell \right) \right\} = \min \left\{ r(\mu_m + \mu_\ell - r), r\mu_\ell \left( \frac{a+1}{a} \right) \right\} \\
&= \min \{ r(\mu_m + \mu_\ell - r), \mu_\ell \}.
\end{aligned}$$

Notice that this formula is also valid at  $m = 0$ . Hence

$$A_{\alpha(s)}(\mu_m, \eta_\ell) = \begin{cases} \max \{ (1-r)\mu_m + r\eta_\ell - r(1-r), 0 \} & \text{if } m < 0; \\ \min \{ r(\mu_m + \eta_\ell - r), \eta_\ell \} & \text{if } m \geq 0, \end{cases}$$

as shown in (4.7). On the other hand, if  $\ell \geq 0$  then  $\eta_{-\ell}^* = 0$  or  $\eta_p$  for some  $p < 0$ , where the previous case can be applied. Now, we compute

$$A_{\alpha(s)}(\mu_{-m}^*, \eta_{-\ell}^*) = \begin{cases} \min \{ r(1 - \mu_m + 1 - \eta_\ell - r), 1 - \eta_\ell \} & \text{if } m < 0; \\ \max \{ (1-r)(1 - \mu_m) + r(1 - \eta_\ell) - r(1-r), 0 \} & \text{if } m \geq 0, \end{cases}$$

$$= \begin{cases} \min \{r(2-r) - r(\mu_m + \eta_\ell), 1 - \eta_\ell\} & \text{if } m < 0; \\ \max \{1 - r(1-r) - (1-r)\mu_m - r\eta_\ell, 0\} & \text{if } m \geq 0. \end{cases}$$

Hence by (6.11),

$$\begin{aligned} A_{\alpha(s)}(\mu_m, \eta_\ell) &= \mu_m + \eta_\ell - 1 + A_{\alpha(s)}(\mu_{-m}^*, \eta_{-\ell}^*) \\ &= \begin{cases} \min \{\mu_m, (1-r)(\mu_m + \eta_\ell - (1-r))\} & \text{if } m < 0; \\ \max \{r\mu_m + (1-r)\eta_\ell - r(1-r), \mu_m + \eta_\ell - 1\} & \text{if } m \geq 0. \end{cases} \end{aligned}$$

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