Countably Piecewise Monotonic Surjection Implicit Dependence Copulas

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Abstract

The copula of two uniform-$(0,1)$ random variables U and V is called *implicit dependence* if there exist measure-preserving transformations α and β on $[0,1]$ such that $\alpha(U) = \beta(V)$ almost surely. Relationships between the implicit dependence copula $C_{U,V}$ and the complete dependence copulas $C_{e,\alpha}$ and $C_{\beta,e}$, where e is the identity map, are investigated. In the case where both α and β are countably piecewise monotonic surjections, we obtain a characterization of implicit dependence copulas in terms of generalized Markov products of $C_{e,\alpha}$ and $C_{\beta,e}$.

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1 Introduction

The most important role copulas play in probabilistic modeling is to describe dependence among random variables up to $\mathcal{U}(0,1)$ marginals. Sifting for a suitable copula can be facilitated by key features of their dependence structures, e.g. asymptotic dependence, tail dependence coefficients, rank correlation coefficients, distance correlation coefficients or probability mass distribution. See for instance [12, 7]. Due partly to the convenience

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of probability density functions, absolutely continuous copulas have been investigated and utilized in practice much earlier than other types of copulas. More deterministic but mostly simpler to manipulate is the singular copulas whose mass is concentrated on a set of Lebesgue measure zero. Typical two-dimensional singular copulas are the copulas of two continuous random variables where one is a Borel function of the other, called the *complete dependence (CD) copulas.* They are always of the form $C_{e,\alpha}$ or $C_{\alpha,e}$ for some measure-preserving function α, where e is the identity function on $[0,1]$ and $C_{\alpha,\beta}(x,y) = \lambda\left(\alpha^{-1}([0,x]) \cap \beta^{-1}([0,y])\right)$. This simple analytic characterization of the CD copulas gives rise to its ubiquity in copula theory [21]. In the class of bivariate copulas, even though the CD copulas form a dense set in the supremum norm, the set is quite small in dependence-indicating norms such as the modified Sobolev norm [19, 16]. Hence, some weaker types of dependence between random variables and the corresponding copulas should be explored further, especially as representatives of copulas sharing the same level of dependence.

Two random variables X and Y are said to be *implicitly dependent* if $f(X) = g(Y)$ almost surely for some Borel functions f and g. See [2, p. 144] and [15, p. 283]. In [20], *implicit dependence copulas*, defined as the copulas of implicitly dependent continuous random variables, was thought speciously to be factorizable as $C_{e\alpha} * C_{\beta e}$. This claim, though not relevant to the results proved in [20], turns out to be false as a simple example can be constructed from $X \sim \mathcal{U}(0,1)$ and $Z \sim Ber(p)$ such that X, Z are independent by setting $Y = \mathbb{1}_{\{1\}}(Z)X + \mathbb{1}_{\{0\}}(Z)(1-X)$. Then $Y \sim \mathcal{U}(0,1)$, $\Lambda_{1/2}(X) = \Lambda_{1/2}(Y)$ and $C_{(X,Y)} = pM + (1-p)W$, which is not factorizable if $p \notin \{0, 1/2, 1\}$, where M, W are the Fréchet-Hoeffding upper and lower bounds and $\Lambda_\theta(x) := \min\left(\frac{x}{\theta}, \frac{1-x}{1-\theta}\right)$ for $\theta \in (0,1)$. As far as we are aware, the first connection between implicit dependence copulas and complete dependence copulas appeared in [13] for a very special case of implicit dependence copulas $C_{U,V}$, where U, V are $\mathcal{U}(0,1)$-random variables such that $\Lambda_\theta(U) = \Lambda_\theta(V)$ a.s., stating that such implicit dependence copulas are exactly the generalized Markov product $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}$ for all $\mathcal{A} = \{A_t\}_{t\in[0,1]}$ such that $t \mapsto A_t(\theta,\theta)$ is measurable in t.

In this paper, we shall generalize the above characterization to the case where $\alpha(U) = \beta(V)$ a.s. for some measure-preserving transformations α and β that are countably piece-wise monotonic surjection. Following the notation in Definition 1, we obtain that a copula C is in $\mathscr{C}_{\text{PID}}$ if and only if there exist $\alpha, \beta \in \mathscr{T}_{\text{P}}$ and $\mathcal{A} := \{A_t\}_{t\in[0,1]} \subseteq \mathscr{C}$ such that $A_t(x,y)$ is measurable in t for all $(x,y) \in \mathbb{I}^2$ and $C = C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$. All requisite background will be given in the next section. The "if" part (Theorem 5) will be discussed in section 3, while the "only if" part (Theorem 10) will be proved in section 4. In section 5, we show via examples how to find $\mathcal{A}$ and factorize checkmin copulas and a copula with hairpin support.

2 Background

For subsets S_1 and S_2 of $\mathbb{I} := [0,1]$ both containing 0 and 1, a function $A\colon S_1 \times S_2 \to [0,1]$ is called a *subcopula* on $S_1 \times S_2$ if $A(0,y) = 0 = A(x,0)$ and $A(1,y) = y$, $A(x,1) = x$ for all $x \in S_1$ and $y \in S_2$ and A is 2-increasing:

$$V_A(B) := A(x_2,y_2) - A(x_2,y_1) - A(x_1,y_2) + A(x_1,y_1) \geq 0$$

for every rectangle $B := [x_1,x_2] \times [y_1,y_2]$ for which all (x_i,y_j) are in $S_1 \times S_2$. A subcopula on $[0,1]^2$ is called a *copula*. It then follows that every copula is non-decreasing in each variable and Lipschitz with respect to the ℓ^1-norm on $\mathbb{I}^2$. This implies that the first partial derivatives of a copula exist almost everywhere with values in $[0,1]$. See [12] for a detailed introduction to copulas.

Let λ denote Lebesgue measure on $\mathbb{I}$ endowed with the Borel σ-algebra $\mathscr{B} := \mathscr{B}(\mathbb{I})$. A Borel transformation f on $\mathbb{I}$ is said to be *measure-preserving* if $\lambda\left(f^{-1}(B)\right) = \lambda(B)$ for every $B \in \mathscr{B}$. The identity on $\mathbb{I}$ is denoted by e. Corresponding to each pair of measure-preserving transformations f and g on $\mathbb{I}$, a copula $C_{f,g}\colon \mathbb{I}^2 \to \mathbb{I}$ is defined by

$$C_{f,g}(x,y) := \lambda\Big(f^{-1}([0,x]) \cap g^{-1}([0,y])\Big).$$

Conversely, every copula can be written as $C = C_{f,g}$ for some measure-preserving transformations f and g. Proofs of this characterization of copulas can be found in [3, 4, 22].

For continuous random variables X and Y, whose joint distribution function is H and marginal distribution functions are F_X and F_Y respectively, the Sklar's theorem says that there exists a unique copula, denoted by $C_{(X,Y)}$, such that $H(x,y) = C_{(X,Y)}\left(F_X(x), F_Y(y)\right)$ for $x, y \in \mathbb{R}$. By the probability integral transform, both $F_X(X)$ and $F_Y(Y)$ are uniformly distributed on $[0,1]$, written as $F_X(X), F_Y(Y) \sim \mathcal{U}(0,1)$, and copulas are simply joint distribution functions of uniformly distributed random variables. As such, it is universal to view the copula of X, Y as containing all their marginal-free dependence structure.

Random variables X and Y are said to be *completely dependent* if there exists a Borel function f such that $Y = f(X)$ a.s. or $X = f(Y)$ a.s.; and they are said to be *implicitly dependent* if there exist Borel functions f and g such that $f(X) = g(Y)$ a.s. If X and Y are completely/implicitly dependent continuous random variables, then their copula $C_{(X,Y)}$ is called a *complete/implicit dependence copula*. It is well known [7, 21] that a copula C is complete dependence if and only if $C = C_{e,\psi}$ or $C = C_{\psi,e}$ for some measure-preserving transformation ψ. In fact, if $U \sim \mathcal{U}(0,1)$ then $C_{(U,\psi(U))} = C_{e,\psi}$ and $C_{(\psi(U),U)} = C_{\psi,e}$. Our goal is to obtain an analogous characterization of implicit dependence copulas.

For convenience, let us denote by $\mathscr{C}$ the class of copulas, $\mathscr{C}_{\text{CD}}$ the class of complete dependence copulas, $\mathscr{C}_{\text{ID}}$ the class of implicit dependence copulas and $\mathscr{T}$ the class of measure-preserving functions on $\mathbb{I}$. In this manuscript, we shall investigate a proper subclass of $\mathscr{C}_{\text{ID}}$ defined as follows.

Definition 1. *A measure-preserving transformation* α *on* $[0,1]$ *is called a* countably piecewise monotonic surjection *(CPMS), written* $\alpha \in \mathscr{T}_P$*, if there is a partition (a.e.)* $P := \{I_n\}_{n\in\mathbb{N}}$ *of* $[0,1]$ *consisting of open intervals* $I_n := (a_n, b_n)$ *such that* $\sum_n (b_n - a_n) = 1$ *and each* $\alpha_n := \alpha|_{I_n}$ *is a strictly monotonic function from* I_n *onto* $(0,1)$*.*

An implicit dependence copula C *is said to be in* $\mathscr{C}_{PID}$ *if there exist random variables* $U, V \sim \mathcal{U}(0,1)$ *and* $\alpha, \beta \in \mathscr{T}_P$ *for which* $C = C_{U,V}$ *and* $\alpha(U) = \beta(V)$ *a.s.*

The *generalized Markov product* of copulas C and D with respect to a parametric class of copulas $\mathcal{A} := \{A_t\}_{t\in[0,1]}$ is formally defined as

$$C *_{\mathcal{A}} D(x,y) := \int_0^1 A_t(\partial_2 C(x,t), \partial_1 D(t,y))dt \quad \text{for } x, y \in [0,1].$$

If the map $(t,x,y) \to A_t(x,y)$ is Borel measurable, then the function $C *_{\mathcal{A}} D$ is a copula. See [5, 6, 17, 7]. The product is simply written $C *_A D$ if $A_t = A$ for all (a.e.) t. And it is the usual *Markov product* on $\mathscr{C}$ if $A_t = \Pi$ for all (a.e.) t.

Let us recall our main tool borrowed from disintegration theorem. See [9, 11]. Given measurable spaces $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$, a mapping $K\colon \Omega_1 \times \mathscr{F}_2 \to \mathbb{R}$ is called a *Markov kernel* if $\omega_1 \mapsto K(\omega_1, B)$ is $\mathscr{F}_1$-measurable for every fixed $B \in \mathscr{F}_2$ and $B \mapsto K(\omega_1, B)$ is a probability measure for every fixed $\omega_1 \in \Omega_1$. For real-valued random variables X, Y, there exists a Markov kernel K, called a *regular conditional distribution* of Y given X, satisfying $K(X(\omega), B) = \mathbb{P}(Y \in B \mid X)(\omega)$ $\mathbb{P}$-a.s. Then, for every Borel function f on $\mathscr{B}(\mathbb{R}^2)$ with $\mathbb{E}[|f(Y,X)|] < \infty$, $\mathbb{E}[f(Y,X) \mid X] = \int_{\mathbb{R}} f(s,X)K(X,ds)$ a.s. Consequently, if C is a copula of random variables $U, V \sim \mathcal{U}(0,1)$ and K_C is a version of the regular conditional distribution of V given U,

$$C(x,y) = \int_0^x K_C(s,[0,y])ds. \tag{2.1}$$

Lastly, we recall some theorems used in approximating the conditional probability given $X = x$ by the conditional probability given X in a neighborhood of x shrinking nicely to x. A sequence $\{E_i\}_{i\in\mathbb{N}}$ of Borel sets in $\mathbb{R}$ is said to *shrink nicely* to $x \in \mathbb{R}$ if there exist a number $\alpha > 0$ and a sequence (r_i) of positive real numbers converging to 0 such that $E_i \subseteq B(x, r_i)$ and $\lambda(E_i) \geq \alpha\lambda(B(x, r_i))$ for all i.

Theorem 2. *([18, Theorem 7.10, p.140]) For each* x *in* $\mathbb{R}$*, let a sequence* $\{E_j(x)\}_{j=1}^{\infty}$ *shrink to* x *nicely and let* $f \in L^1(\mathbb{R})$*. Then, at almost everywhere* x*,*

$$f(x) = \lim_{j\to\infty} \frac{1}{\lambda(E_j(x))} \int_{E_j(x)} f\, d\lambda.$$

Applying Theorem 2 to $f(x) = \mathbb{P}(Y \in A \mid X = x)$ and using the identity [1, Theorem 5.3.1, p.205] $\int_{E_j} \mathbb{P}(Y \in A \mid X = t) d\mathbb{P}_X(t) = \mathbb{P}(Y \in A, X \in E_j)$, we have the following.

Theorem 3. *Let X and Y be random variables, $A \in \mathscr{B}(\mathbb{R})$ and $\{E_j(x)\}_{j=1}^{\infty}$ a sequence that shrinks to $x \in \mathbb{R}$ nicely. Then*

$$\mathbb{P}(Y \in A \mid X = x) = \lim_{j\to\infty} \mathbb{P}\left(Y \in A \mid X \in E_j(x)\right).$$

3 Generalized products of two complete dependence copulas

Let α and β be CPMS measure-preserving transformations on $[0,1]$, i.e. $\alpha, \beta \in \mathscr{T}_{\mathrm{P}}$, with partitions $P := \{I_n := (a_n, b_n)\}_{n\in\mathbb{N}}$ and $Q := \{J_n := (c_n, d_n)\}_{n\in\mathbb{N}}$, respectively. Denote $\alpha_n := \alpha|_{I_n}$ and $\beta_n := \beta|_{J_n}$. By [10], $\alpha_{ij} := \alpha_i^{-1} \circ \alpha_j, \beta_{ij} := \beta_i^{-1} \circ \beta_j$ and $\gamma_{ij} := \beta_i^{-1} \circ \alpha_j$ are strictly monotonic Borel measurable functions from I_j onto I_i, J_j onto J_i, and I_j onto J_i, respectively. Consequently, all functions above are differentiable with nonzero derivatives a.e. on their domains [8]. To refer to points outside the domains, we shall denote $\partial P := [0,1] \setminus \bigcup_{n=1}^{\infty} I_n$ and $\partial Q := [0,1] \setminus \bigcup_{n=1}^{\infty} J_n$. Clearly, $\lambda(\partial P) = \lambda(\partial Q) = 0$.

Remark 4.

1. *For $m, n \in \mathbb{N}$, we define a total order $\preceq$ on P by $I_m \preceq I_n$ if and only if $a_m \le a_n$. For convenience, we say that $I_m \prec I_n$ if $a_m < a_n$. Similar total order and relation on the partition Q are defined and also denoted by $\preceq$ and $\prec$, respectively.*

2. *For $s \in [0,1]\backslash\partial P$ ($t \in [0,1]\backslash\partial Q$) and $i \in \mathbb{N}$, let $s_{(i)}$ $\left(t^{(i)}\right)$ denote the unique number in I_i (J_i) such that $\alpha(s_{(i)}) = \alpha(s)$ $\left(\beta(t^{(i)}) = \beta(t)\right)$. Obviously, $s = s_{(i)}$ $\left(t = t^{(i)}\right)$ if and only if $s \in I_i$ ($t \in J_i$). Consequently, if $s \in I_k$ ($t \in J_k$), then $s_{(i)} = \alpha_{ik}(s)$ $\left(t^{(i)} = \beta_{ik}(t)\right)$. Since every α_i (β_i) is strictly monotonic, it holds that for almost every $u, v \in [0,1]$,*

   - *if α_i and α_j (β_i and β_j) are either both increasing or both decreasing, then $u_{(i)} < v_{(i)}$ $\left(u^{(i)} < v^{(i)}\right)$ is equivalent to $u_{(j)} < v_{(j)}$ $\left(u^{(j)} < v^{(j)}\right)$;*
   - *otherwise, $u_{(i)} < v_{(i)}$ $\left(u^{(i)} < v^{(i)}\right)$ is equivalent to $u_{(j)} > v_{(j)}$ $\left(u^{(j)} > v^{(j)}\right)$.*

   Moreover, $\alpha^{-1}\left((0, \alpha(s)]\right) = \left(\bigcup_{i\in I_{inc}} (a_i, s_{(i)}]\right) \cup \left(\bigcup_{i\in I_{dec}} [s_{(i)}, b_i)\right)$ where

   $$I_{inc} := \{i \in \mathbb{N} : \alpha_i \text{ is increasing}\} \quad \text{and} \quad I_{dec} := \mathbb{N}\backslash I_{inc}.$$

   Employing β_i's in place of α_i's, the partition $\{J_{inc}, J_{dec}\}$ of $\mathbb{N}$ can be defined in a similar manner resulting in analogous statements for $\beta^{-1}\left((0, \beta(t)]\right)$.

3. *For almost every $x, y \in [0,1]$ and almost every $s, t \in (0,1)$ such that $\alpha(s) = \beta(t)$, we have $x = x_{(k)}$, $y = y^{(\ell)}$ for some $k, \ell \in \mathbb{N}$, $\partial_2 C_{e,\alpha}(x_{(k)}, \alpha(s)) = \displaystyle\sum_{i=1}^{\infty} \frac{1}{|\alpha'(s_{(i)})|} \mathbb{1}_{(0,x_{(k)})}(s_{(i)})$*

and $\partial_2 C_{e,\beta}(y^{(\ell)}, \alpha(s)) = \partial_2 C_{e,\beta}(y^{(\ell)}, \beta(t)) = \sum_{j=1}^{\infty} \frac{1}{|\beta'(t^{(j)})|} \mathbb{1}_{(0,y^{(\ell)})}(t^{(j)})$. *Equivalently,*

$$\partial_2 C_{e,\alpha}(x_{(k)}, \alpha(s)) = \begin{cases} \mu_k^* & \text{if } x_{(k)} < s_{(k)}, \\ \mu_k & \text{if } x_{(k)} > s_{(k)}, \end{cases} \tag{3.1}$$

where $\mu_k := \sum_{I_i \preceq I_k} \frac{1}{|\alpha'(s_{(i)})|}$ *and* $\mu_k^* := \sum_{I_i \prec I_k} \frac{1}{|\alpha'(s_{(i)})|} = \mu_k - \frac{1}{|\alpha'(s_{(k)})|}$; *and likewise for* $\partial_2 C_{e,\beta}$. *I.e.,* $\partial_2 C_{e,\beta}(y^{(\ell)}, \alpha(s)) = \eta_\ell \mathbb{1}_{[0,y^{(\ell)})}(t^{(\ell)}) + \eta_\ell^* \mathbb{1}_{(y^{(\ell)},1]}(t^{(\ell)})$ *where* $\eta_\ell := \sum_{J_j \preceq J_\ell} \frac{1}{|\beta'(t^{(j)})|}$ *and* $\eta_\ell^* := \sum_{J_j \prec J_\ell} \frac{1}{|\beta'(t^{(j)})|} = \eta_\ell - \frac{1}{|\beta'(t^{(\ell)})|}$. *All summations are convergent because they are all bounded by one. Moreover, by the facts that all summands are positive and that the collection* $\mathscr{I}$ *of indexed sets* $\{i \colon I_i \preceq I_k\}$ *and* $\{i \colon I_i \prec I_k\}$, $k \in \mathbb{N}$, *is totally ordered, for every value* μ *in* $\{\mu_k, \mu_k^* \colon k \in \mathbb{N}\}$ *there exists a unique* $\mathcal{I}_\mu \in \mathscr{I}$ *such that* $\mu = \sum_{i \in \mathcal{I}_\mu} \frac{1}{|\alpha'(s_{(i)})|}$. *Similarly, for every* η *in* $\{\eta_\ell, \eta_\ell^* \colon \ell \in \mathbb{N}\}$ *there corresponds a unique indexed set* $\mathcal{J}_\eta$ *of the form* $\{j \colon J_j \preceq J_\ell\}$ *or* $\{j \colon J_j \prec J_\ell\}$, *for some* $\ell \in \mathbb{N}$, *such that* $\eta = \sum_{j \in \mathcal{J}_\eta} \frac{1}{|\beta'(t^{(j)})|}$.

To prove (3.1), *let* $F(k,z) := \sum_{\substack{i \in I_{inc} \\ I_i \prec I_k}} \left(\alpha_i^{-1}(z) - a_i\right) + \sum_{\substack{i \in I_{dec} \\ I_i \prec I_k}} \left(b_i - \alpha_i^{-1}(z)\right)$ *and observe from the definition of* $C_{e,\alpha}$ *that*

$$C_{e,\alpha}\left(x_{(k)}, \alpha(s)\right) = \begin{cases} F(k, \alpha(s)) + \min\left(x_{(k)}, s_{(k)}\right) - a_k & \text{if } k \in I_{inc}, \\ F(k, \alpha(s)) + \max\left(x_{(k)} - s_{(k)}, 0\right) & \text{if } k \in I_{dec}. \end{cases}$$

For $s \in \mathbb{I} \setminus \partial P$ *and* h *small enough that* $\alpha(s) + h \in (0,1)$, *it is evident that*

$$\frac{F(k, \alpha(s)+h) - F(k, \alpha(s))}{h} = \sum_{I_i \prec I_k} \left| \frac{\alpha_i^{-1}(\alpha(s)+h) - \alpha_i^{-1}(\alpha(s))}{h} \right|$$

and hence $\frac{C_{e,\alpha}(x_{(k)}, \alpha(s)+h) - C_{e,\alpha}(x_{(k)}, \alpha(s))}{h}$ *is equal to*

$$\begin{cases} \sum_{I_i \prec I_k} \left| \frac{\alpha_i^{-1}(\alpha(s)+h) - \alpha_i^{-1}(\alpha(s))}{h} \right| & \text{if } x_{(k)} < s_{(k)}, \\ \sum_{I_i \preceq I_k} \left| \frac{\alpha_i^{-1}(\alpha(s)+h) - \alpha_i^{-1}(\alpha(s))}{h} \right| & \text{if } x_{(k)} > s_{(k)}. \end{cases}$$

For $s = s_{(i)} \in I_i$ *at which* α_i *is differentiable,* α_i^{-1} *is differentiable at* $\alpha(s)$ *and*

$$\lim_{h \to 0} \frac{\alpha_i^{-1}(\alpha(s)+h) - \alpha_i^{-1}(\alpha(s))}{h} = \left(\alpha_i^{-1}\right)'(\alpha(s)) = \frac{1}{\alpha_i'(s)}.$$

It then follows straightforwardly from the definition of $\partial_2 C_{e,\alpha}(x, \alpha(s))$ *that* (3.1) *holds a.e.* x *and* s. *The second equation can be derived analogously.*

4. *For $y \in \mathbb{I}$, $K_C(s,[0,y]) = K_C(s,[0,y]\backslash \partial Q)$ a.e. s because $\mathbb{P}(Y \in \partial Q \mid X) = 0$ a.s.*

5. *The case that $P := \{I_i\}_{i=1}^m$ or $Q := \{J_j\}_{j=1}^n$ are finite partitions on $[0,1]$ can be viewed as a special case of our work by letting $I_i = \emptyset$ for $i > m$ or $J_j = \emptyset$ for $j > n$, respectively.*

Theorem 5. *Let $\mathcal{A} := \{A_t\}_{t\in[0,1]}$ be a class of copulas such that $A_t(x,y)$ is Borel measurable in t for all $(x,y) \in \mathbb{I}^2$. For CPMS measure-preserving transformations α, β, the copula $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ is an implicit dependence copula. In fact, there exist random variables X and Y uniformly distributed on $[0,1]$ such that $\alpha(X) = \beta(Y)$ a.s. and $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e} = C_{(X,Y)}$.*

Proof. At every $(x,y) \in \mathbb{I}^2$, $\partial_2 C_{e,\alpha}(x,t)$ and $\partial_2 C_{e,\beta}(y,t)$ are measurable in t by Lebesgue differentiation theorem; and so $A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t))$ is measurable in t. Consequently, $C' := C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$ is well-defined and hence a copula. Let $\mu_{C'}$ be the Borel probability measure extension of $V_{C'}$ to $\mathscr{B}(\mathbb{I}^2)$. Define random variables X and Y as the projection maps of $\mathbb{I}^2$ onto the x- and y-coordinates, respectively. Since $\mu_{C'}$ has uniform marginals, X and Y are uniformly distributed on $[0,1]$. Also, their joint distribution function is C', i.e. $C_{(X,Y)} = C'$. It then remains to show that $\alpha \circ X(x,y) = \beta \circ Y(x,y)$ for $\mu_{C'}$-a.e. $(x,y) \in \mathbb{I}^2$, which will be shown on each $A_{ij} := I_i \times J_j$ that $V_{C'}(A_{ij} \setminus G) = 0$ where $G := \{\alpha(X) = \beta(Y)\} = \{(x,y) \in \mathbb{I}^2 : y = \gamma_{ji}(x) \text{ for some } i,j\}$. By the monotonicity of γ_{ji}, every rectangular subset of $A_{ij}\backslash G$ can be written as a combination of no more than four rectangles in $A_{ij}\backslash G$ all of which have two sides lying on the boundary of A_{ij}. It then suffices to show that $V_{C'}(B) = 0$ whenever B is such a rectangle.

For almost every $(x,y) \in I_p \times J_q$ and almost every $t \in (0,1)$, (3.1) for both $C_{e,\alpha}$ and $C_{e,\beta}$ can be rewritten simply as combinations of indicator functions, of which the coefficients depend on whether α_p and β_q are increasing or decreasing as follows:

$$
\begin{aligned}
\partial_2 C_{e,\alpha}(x,t) &= \hat{\mu}_p \mathbb{1}_{[0,\alpha(x))}(t) + \check{\mu}_p \mathbb{1}_{(\alpha(x),1]}(t) \quad \text{ with } \{\hat{\mu}_p, \check{\mu}_p\} = \{\mu_p, \mu_p^*\} \text{ and} \\
\partial_2 C_{e,\beta}(y,t) &= \hat{\eta}_q \mathbb{1}_{[0,\beta(y))}(t) + \check{\eta}_q \mathbb{1}_{(\beta(y),1]}(t) \quad \text{ with } \{\hat{\eta}_q, \check{\eta}_q\} = \{\eta_q, \eta_q^*\},
\end{aligned}
$$

where $\hat{\mu}_p = \mu_p$ if and only if α_p is increasing; and $\hat{\eta}_q = \eta_q$ if and only if β_q is increasing. In particular, $\partial_2 C_{e,\alpha}(a_p,t) = \mu_p^*$, $\partial_2 C_{e,\alpha}(b_p,t) = \mu_p$, $\partial_2 C_{e,\beta}(c_q,t) = \eta_q^*$ and $\partial_2 C_{e,\beta}(d_q,t) = \eta_q$. The symbols μ_p, μ_p^*, η_q and η_q^* were defined in Remark 4(3). Before considering the cases where $\gamma_{qp} := \beta_q^{-1} \circ \alpha_p$ is increasing and decreasing, let us note here that for a.e. $(x,y) \in I_p \times J_q$,

$$
A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t)) = \begin{cases} A_t(\hat{\mu}_p, \hat{\eta}_q) & \text{if } t < M_1, \\ A_t(\check{\mu}_p, \check{\eta}_q) & \text{if } t > M_2, \\ A_t(\check{\mu}_p, \hat{\eta}_q) & \text{if } \alpha(x) < t < \beta(y), \\ A_t(\hat{\mu}_p, \check{\eta}_q) & \text{if } \beta(y) < t < \alpha(x), \end{cases} \tag{3.2}
$$

where $M_1 := \min\{\alpha(x), \beta(y)\}$ and $M_2 := \max\{\alpha(x), \beta(y)\}$.

Case 1: γ_{qp} is increasing.

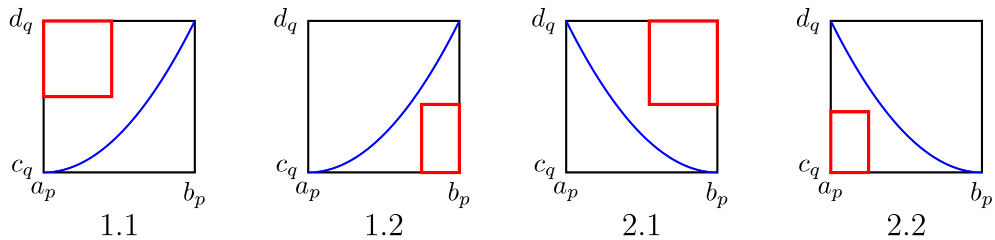

Figure 1: Possible rectangles of the four subcases in Theorem 5

Subcase 1.1 $B = [a_p, x] \times [y, d_q]$ where $y > \gamma_{qp}(x)$. Then $V_{C'}(B)$ equals

$$\begin{aligned}
&C'(x, d_q) - C'(a_p, d_q) - C'(x, y) + C'(a_p, y)\\
&= \left[\int_0^{\alpha(x)} A_t\left(\hat{\mu}_p, \eta_q\right) \mathrm{d}t + \int_{\alpha(x)}^1 A_t\left(\check{\mu}_p, \eta_q\right) \mathrm{d}t\right] - \left[\int_0^1 A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t\right]\\
&\quad - \left[\int_0^{M_1} A_t\left(\hat{\mu}_p, \hat{\eta}_q\right) \mathrm{d}t + \int_{M_1}^{M_2} A_t\left(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t)\right) \mathrm{d}t + \int_{M_2}^1 A_t\left(\check{\mu}_p, \check{\eta}_q\right) \mathrm{d}t\right]\\
&\quad + \left[\int_0^{\beta(y)} A_t\left(\mu_p^*, \hat{\eta}_q\right) \mathrm{d}t + \int_{\beta(y)}^1 A_t\left(\mu_p^*, \check{\eta}_q\right) \mathrm{d}t\right].
\end{aligned}$$

- If β_q is increasing, then so is α_p. Thus, $\hat{\mu}_p = \mu_p$ and $\hat{\eta}_q = \eta_q$, which yields $\check{\mu}_p = \mu_p^*$ and $\check{\eta}_q = \eta_q^*$. In this case, we also have $M_1 = \alpha(x)$, $M_2 = \beta(y)$ and the above expression of $V_{C'}(B)$ reduces to

$$\int_{\alpha(x)}^1 A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t - \int_0^1 A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t - \int_{\alpha(x)}^{\beta(y)} A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t + \int_0^{\beta(y)} A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t = 0.$$

- If β_q is decreasing, then α_p is decreasing. Hence, $\hat{\mu}_p = \mu_p^*$ and $\hat{\eta}_q = \eta_q^*$, which gives $\check{\mu}_p = \mu_p$ and $\check{\eta}_q = \eta_q$. Also $M_1 = \beta(y)$, $M_2 = \alpha(x)$, and the above expression of $V_{C'}(B)$ reduces to

$$\int_0^{\alpha(x)} A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t - \int_0^1 A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t - \int_{\beta(y)}^{\alpha(x)} A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t + \int_{\beta(y)}^1 A_t\left(\mu_p^*, \eta_q\right) \mathrm{d}t = 0.$$

Subcase 1.2 $B = [x, b_p] \times [c_q, y]$ where $y < \gamma_{qp}(x)$. Then $V_{C'}(B)$ equals

$$\begin{aligned}
&C'(b_p, y) - C'(x, y) - C'(b_p, c_q) + C'(x, c_q)\\
&= \left[\int_0^{\beta(y)} A_t\left(\mu_p, \hat{\eta}_q\right) \mathrm{d}t + \int_{\beta(y)}^1 A_t\left(\mu_p, \check{\eta}_q\right) \mathrm{d}t\right]\\
&\quad - \left[\int_0^{M_1} A_t\left(\hat{\mu}_p, \hat{\eta}_q\right) \mathrm{d}t + \int_{M_1}^{M_2} A_t\left(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t)\right) \mathrm{d}t + \int_{M_2}^1 A_t\left(\check{\mu}_p, \check{\eta}_q\right) \mathrm{d}t\right]\\
&\quad - \left[\int_0^1 A_t\left(\mu_p, \eta_q^*\right) \mathrm{d}t\right] + \left[\int_0^{\alpha(x)} A_t\left(\hat{\mu}_p, \eta_q^*\right) \mathrm{d}t + \int_{\alpha(x)}^1 A_t\left(\check{\mu}_p, \eta_q^*\right) \mathrm{d}t\right].
\end{aligned}$$

By a similar argument to subcase 1.1, we obtain $V_{C'}(B) = 0$.

Case 2: γ_{qp} is decreasing.

Subcase 2.1 $B = [x, b_p] \times [y, d_q]$ where $y > \gamma_{qp}(x)$. Then $V_{C'}(B)$ equals

$$\begin{aligned}
&C'(b_p, d_q) - C'(x, d_q) - C'(b_p, y) + C'(x, y) \\
&= \left[\int_0^1 A_t(\mu_p, \eta_q)\,\mathrm{d}t\right] - \left[\int_0^{\alpha(x)} A_t(\hat{\mu}_p, \eta_q)\,\mathrm{d}t + \int_{\alpha(x)}^1 A_t(\check{\mu}_p, \eta_q)\,\mathrm{d}t\right] \\
&\quad - \left[\int_0^{\beta(y)} A_t(\mu_p, \hat{\eta}_q)\,\mathrm{d}t + \int_{\beta(y)}^1 A_t(\mu_p, \check{\eta}_q)\,\mathrm{d}t\right] \\
&\quad + \left[\int_0^{M_1} A_t(\hat{\mu}_p, \hat{\eta}_q)\,\mathrm{d}t + \int_{M_1}^{M_2} A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t))\,\mathrm{d}t + \int_{M_2}^1 A_t(\check{\mu}_p, \check{\eta}_q)\,\mathrm{d}t\right],
\end{aligned}$$

which can be shown to be equal to 0 by considering the cases where β_q is increasing or decreasing in the same manner as in subcase 1.1.

Subcase 2.2 $B = [a_p, x] \times [c_q, y]$ where $y < \gamma_{qp}(x)$. Then $V_{C'}(B)$ equals

$$\begin{aligned}
&C'(x, y) - C'(a_p, y) - C'(x, c_q) + C'(a_p, c_q) \\
&= \left[\int_0^{M_1} A_t(\hat{\mu}_p, \hat{\eta}_q)\,\mathrm{d}t + \int_{M_1}^{M_2} A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t))\,\mathrm{d}t + \int_{M_2}^1 A_t(\check{\mu}_p, \check{\eta}_q)\,\mathrm{d}t\right] \\
&\quad - \left[\int_0^{\beta(y)} A_t\left(\mu_p^*, \hat{\eta}_q\right)\mathrm{d}t + \int_{\beta(y)}^1 A_t\left(\mu_p^*, \check{\eta}_q\right)\mathrm{d}t\right] \\
&\quad - \left[\int_0^{\alpha(x)} A_t\left(\hat{\mu}_p, \eta_q^*\right)\mathrm{d}t + \int_{\alpha(x)}^1 A_t\left(\check{\mu}_p, \eta_q^*\right)\mathrm{d}t\right] + \left[\int_0^1 A_t\left(\mu_p^*, \eta_q^*\right)\mathrm{d}t\right],
\end{aligned}$$

which is equal to 0 by the same argument. □

4 Generalized factorizability of implicit dependence copulas

In this section, the implicit dependence copula of random variables $X, Y \sim \mathcal{U}(0,1)$, where $\alpha(X) = \beta(Y)$ a.s. for some CPMS measure-preserving transformations α, β (i.e. $\alpha, \beta \in \mathscr{T}_{\mathrm{P}}$), will be written as the product of two complete dependence copulas.

Let $\{I_n\}$ and $\{J_n\}$ be the partitions of α and β, respectively. For each $i, j \in \mathbb{N}$, denote $A_{ij} := \{X \in I_i, Y \in J_j\}$ on which $\alpha_i(X) = \beta_j(Y)$ a.s. Hence $Y = \gamma_{ji}(X)$ a.s. on A_{ij} and

$$Y = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \gamma_{ji}(X) \mathbb{1}_{A_{ij}} \quad \text{a.s.} \tag{4.1}$$

Lemma 6. *Let $\alpha, \beta \in \mathscr{T}_P$ with the partitions $\{I_n\}$ and $\{J_n\}$, respectively, and C the copula of $X, Y \sim \mathcal{U}(0,1)$ for which $\alpha(X) = \beta(Y)$ a.s. Let $i, k \in \mathbb{N}$, $y \in (0,1)$ and B a Borel subset of $(0,1]$. Then for a.e. s,* *1.* $K_C(s,B) = \sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} K_C(s,J_j)\mathbb{1}_{\gamma_{j\ell}^{-1}(B)\cap I_\ell}(s)$*;*

2. putting $y^{(k)} := \gamma_{ki}(x_{(i)})$*, we have that*

$$K_C\left(s_{(i)},\left(c_k,y^{(k)}\right]\right) = K_C\left(s_{(i)},J_k\right)\mathbb{1}_{\gamma_{ki}^{-1}\left(c_k,y^{(k)}\right]}\left(s_{(i)}\right)$$
$$= \begin{cases} K_C\left(s_{(i)},J_k\right)\mathbb{1}_{\left(a_i,x_{(i)}\right]}\left(s_{(i)}\right) & \text{if } \gamma_{ki} \text{ is increasing,}\\ K_C\left(s_{(i)},J_k\right)\mathbb{1}_{\left[x_{(i)},b_i\right)}\left(s_{(i)}\right) & \text{if } \gamma_{ki} \text{ is decreasing;}\end{cases}$$

3. setting $y^{(k)} := \gamma_{ki}(x_{(i)})$*, it follows from 2 that*

$$\text{if } \gamma_{ki} \text{ is increasing, then } K_C\left(s_{(i)},\left(0,y^{(k)}\right]\right) = \begin{cases} \sum_{J_j\preceq J_k} K_C(s_{(i)},J_j) & \text{if } s_{(i)} \le x_{(i)},\\ \sum_{J_j\prec J_k} K_C(s_{(i)},J_j) & \text{if } s_{(i)} > x_{(i)};\end{cases}$$

$$\text{if } \gamma_{ki} \text{ is decreasing, then } K_C\left(s_{(i)},\left(0,y^{(k)}\right]\right) = \begin{cases} \sum_{J_j\prec J_k} K_C(s_{(i)},J_j) & \text{if } s_{(i)} < x_{(i)},\\ \sum_{J_j\preceq J_k} K_C(s_{(i)},J_j) & \text{if } s_{(i)} \ge x_{(i)};\end{cases}$$

4. if $\gamma_{ki}(s_{(i)}) \le y^{(k)}$*, then* $K_C\left(s_{(i)},(0,y^{(k)}]\right) = K_C\left(s_{(i)},\bigcup_{J_j\preceq J_k}(0,\gamma_{ji}(s_{(i)})]\right)$*;*

if $\gamma_{ki}(s_{(i)}) > y^{(k)}$*, then* $K_C\left(s_{(i)},(0,y^{(k)}]\right) = K_C\left(s_{(i)},\bigcup_{J_j\prec J_k}(0,\gamma_{ji}(s_{(i)})]\right)$*.*

Here, an empty sum is set to be zero.

Proof. 1. From (4.1), it holds that

$$\mathbb{1}_B\circ Y = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left(\mathbb{1}_{\gamma_{ji}^{-1}(B)}(X)\right)\mathbb{1}_{A_{ij}} \quad \text{a.s.}$$

where we have used the observation that $\mathbb{1}_B\circ(Z\mathbb{1}_{A_{ij}}) = (\mathbb{1}_B\circ Z)\mathbb{1}_{A_{ij}}$ for every i,j and random variable Z taking values in $[0,1]$. Since $\mathbb{1}_{\gamma_{ji}^{-1}(B)}\circ X$ and $\mathbb{1}_{I_i}\circ X$ are $\sigma(X)$-measurable, we have

$$\begin{aligned}\mathbb{E}\left[\left(\mathbb{1}_{\gamma_{ji}^{-1}(B)}\circ X\right)\mathbb{1}_{A_{ij}} \mid X\right] &= \mathbb{E}\left[\left(\mathbb{1}_{\gamma_{ji}^{-1}(B)}\circ X\right)\mathbb{1}_{I_i}(X)\mathbb{1}_{J_j}(Y) \mid X\right]\\ &= \left(\mathbb{1}_{\gamma_{ji}^{-1}(B)\cap I_i}\circ X\right)\mathbb{E}\left[\mathbb{1}_{\{Y\in J_j\}} \mid X\right]\\ &= K_C(X,J_j)\left(\mathbb{1}_{\gamma_{ji}^{-1}(B)\cap I_i}\circ X\right) \quad \text{a.s.}\end{aligned}$$

Thus, $K_C(s,B) = \mathbb{E}\left[\mathbb{1}_B\circ Y \mid X=s\right] = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} K_C(s,J_j)\mathbb{1}_{\gamma_{ji}^{-1}(B)\cap I_i}(s)$ a.e. s.

2. Let $y \in J_k$ and $s \in I_i$ so that $y = y^{(k)}$ and $s = s_{(i)}$. Then $K_C(s,(c_k,y]) = K_C(s,J_k)\,\mathbb{1}_{\gamma_{ki}^{-1}((c_k,y])}(s) = \begin{cases} K_C(s,J_k)\,\mathbb{1}_{(a_i,x_{(i)}]}(s) & \text{if } \gamma_{ki} \text{ is increasing,} \\ K_C(s,J_k)\,\mathbb{1}_{[x_{(i)},b_i)}(s) & \text{if } \gamma_{ki} \text{ is decreasing.} \end{cases}$

3. $K_C\left(s_{(i)},\left(0,y^{(k)}\right]\right) = \sum_{J_j \prec J_k} K_C\left(s_{(i)},J_j\right) + K_C\left(s_{(i)},\left(c_k,y^{(k)}\right]\right)$ and 2 gives the desired result.

4. If $\gamma_{ki}(s_{(i)}) \leq y^{(k)}$ then we have $s_{(i)} \leq \gamma_{ki}^{-1}(y^{(k)})$ or $s_{(i)} \geq \gamma_{ki}^{-1}(y^{(k)})$ provided that γ_{ki} is increasing or decreasing, respectively. This implies by 3 that

$$K_C\left(s_{(i)},(0,y^{(k)}]\right) = \sum_{J_j \preceq J_k} K_C\left(s_{(i)},J_j\right) = K_C\left(s_{(i)},\left(0,\gamma_{ki}(s_{(i)})\right]\right)$$
$$= K_C\left(s_{(i)},\bigcup_{J_j \preceq J_k}(0,\gamma_{ji}(s_{(i)})]\right).$$

The latter statement can be shown similarly. □

Lemma 7. *Let random variables X and Y be uniformly distributed on $[0,1]$ such that $\alpha(X) = \beta(Y)$ a.s. and C the copula of X,Y for some $\alpha,\beta \in \mathscr{T}_P$. Then, for $k \in \mathbb{N}$ and a.e. $s \in [0,1]$,*

$$\sum_{i=1}^{\infty} \frac{1}{|\alpha'(s_{(i)})|} K_C(s_{(i)},J_k) = \frac{1}{|\beta'(r^{(k)})|} \quad \textit{where } \alpha(s) = \beta(r). \tag{4.2}$$

Proof. Let $A_n(t) = \left(t - \frac{1}{n}, t + \frac{1}{n}\right) \cap [0,1]$, where $n \in \mathbb{N}$, which shrinks to $t = \alpha(s)$ nicely. Since $\alpha \in \mathscr{T}_\mathrm{P}$, the event $B_n(t) := \{\alpha(X) \in A_n(t)\}$ can be decomposed as $\dot\bigcup_{i=1}^{\infty}\left\{X \in \alpha_i^{-1}(A_n(t))\right\}$ and by Theorem 3, for a.e. t,

$$\begin{aligned}
\mathbb{P}(Y \in J_k \mid \alpha(X) = t) &= \lim_{n\to\infty} \mathbb{P}\left(Y \in J_k \mid B_n(t)\right) \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{\infty} \frac{\mathbb{P}(Y \in J_k, X \in \alpha_i^{-1}(A_n(t))}{\mathbb{P}(B_n(t))}\right] \\
&= \sum_{i=1}^{\infty} \lim_{n\to\infty}\left[\mathbb{P}(Y \in J_k \mid X \in \alpha_i^{-1}(A_n(t))) \cdot \frac{\mathbb{P}(X \in \alpha_i^{-1}(A_n(t)))}{\mathbb{P}(B_n(t))}\right].
\end{aligned}$$

Since $X \sim \mathcal{U}(0,1)$ and α is measure-preserving, $\mathbb{P}\left(X \in \alpha_i^{-1}(A_n(t))\right) = \lambda\left(\alpha_i^{-1}(A_n(t))\right)$ and $\mathbb{P}\left(B_n(t)\right) = \lambda\left(\alpha^{-1}(A_n(t))\right) = \lambda\left(A_n(t)\right)$. Therefore, by Theorem 2, for a.e. t,

$$\lim_{n\to\infty} \frac{\mathbb{P}\left(X \in \alpha_i^{-1}(A_n(t))\right)}{\mathbb{P}\left(B_n(t)\right)} = \left|\left(\alpha_i^{-1}\right)'(t)\right| = \frac{1}{\left|\alpha_i'\left(s_{(i)}\right)\right|} = \frac{1}{\left|\alpha'\left(s_{(i)}\right)\right|}.$$

As the sequence $\alpha_i^{-1}(A_n(t))$ shrinks nicely to $s_{(i)}$ as $n \to \infty$, we have by Theorem 3 that $\mathbb{P}(Y \in J_k \mid X \in \alpha_i^{-1}(A_n(t))) \to \mathbb{P}(Y \in J_k \mid X = s_{(i)})$. Hence, $\mathbb{P}\left(Y \in J_k \mid \alpha(X) = t\right) =$

$\sum_{i=1}^{\infty} \frac{\mathbb{P}\left(Y \in J_k \mid X = s_{(i)}\right)}{\left|\alpha'\left(s_{(i)}\right)\right|}$. Since $\alpha(X) = \beta(Y)$ a.s. and $\beta(r) = \alpha(s) = t$, we have

$$\mathbb{P}\left(Y \in J_k \mid \alpha(X) = t\right) = \mathbb{P}\left(Y \in J_k \mid \beta(Y) = t\right) = \sum_{j=1}^{\infty} \frac{\mathbb{P}\left(Y \in J_k \mid Y = r^{(j)}\right)}{\left|\beta'\left(r^{(j)}\right)\right|}$$

where a similar argument as above was used in the second equality. Finally, as it is clear that $\mathbb{P}\left(Y \in J_k \mid Y = r^{(j)}\right) = \delta_{jk}$, we obtain $\sum_{i=1}^{\infty} \frac{1}{|\alpha'(s_{(i)})|} \mathbb{P}\left(Y \in J_k \mid X = s_{(i)}\right) = \frac{1}{|\beta'(r^{(k)})|}$ as desired. □

For almost all $s \in [0,1]$, we let $t \in [0,1]$ be such that $\alpha(s) = \beta(t)$. Also, we define $P(s) := \{0,1\} \cup \{\mu_n, \mu_n^*\}_{n\in\mathbb{N}}$ and $Q(s) := \{0,1\} \cup \{\eta_n, \eta_n^*\}_{n\in\mathbb{N}}$.

Lemma 8. *Let C be the copula of $\mathcal{U}(0,1)$-random variables X, Y such that $\alpha(X) = \beta(Y)$ a.s. for some $\alpha, \beta \in \mathscr{T}_P$. For each $s \in I_1$, define the function $A_{\alpha(s)}$ on $P(s) \times Q(s)$ as follows. If $u \in \{0,1\}$ or $v \in \{0,1\}$ then we set $A_{\alpha(s)}(u,v) := u \cdot v$. Otherwise, we define*

$$A_{\alpha(s)}\left(u,v\right) := \frac{1}{|\alpha'(s)|} \sum_{i\in\mathcal{I}_u} |\alpha'_{i1}(s)| K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_v} [0, \gamma_{j1}(s)]\right). \tag{4.3}$$

Then $A_{\alpha(s)}$ is a subcopula on $P(s) \times Q(s)$. By bilinear interpolation, each subcopula $A_{\alpha(s)}$ can be extended to a copula. Also, by setting A_0 and A_1 to be any copulas, we have constructed a collection $\mathcal{A} := \{A_t\}_{t\in[0,1]}$ of copulas associated with a given implicit dependence copula $C = C_{(X,Y)}$ for which $X, Y \sim \mathcal{U}(0,1)$ and $\alpha(X) = \beta(Y)$ a.s.

Recall the definitions of $\mathcal{I}_u$ and $\mathcal{J}_v$ from Remark 4(3) with the border cases defined by $\mathcal{I}_0 = \mathcal{J}_0 := \varnothing$ and $\mathcal{I}_1 = \mathcal{J}_1 := \mathbb{N}$. Then the equation (4.3) also holds for the case $u = 0$ or $v = 0$ if we adopt the convention that an empty sum is 0 and an empty union is $\varnothing$, respectively. In addition, for the case $u = 1$ or $v = 1$, the equation (4.3) still holds using Lemma 7 and Lemma 6(4), respectively.

Proof. It remains to show that $A_{\alpha(s)}$ has 2-increasing property. Let $u_1, u_2 \in P(s)$ and $v_1, v_2 \in Q(s)$ be such that $u_1 \le u_2$ and $v_1 \le v_2$. For $\ell = 1,2$, $A_{\alpha(s)}(u_2, v_\ell) - A_{\alpha(s)}(u_1, v_\ell) = \frac{1}{|\alpha'(s)|} \sum_{i\in\mathcal{I}_{u_2}\setminus\mathcal{I}_{u_1}} |\alpha'_{i1}(s)| K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{v_\ell}} [0, \gamma_{j1}(s)]\right)$. Thus,

$$V_{A_{\alpha(s)}}\left([u_1,u_2] \times [v_1,v_2]\right) = \frac{1}{|\alpha'(s)|} \sum_{i\in\mathcal{I}_{u_2}\setminus\mathcal{I}_{u_1}} |\alpha'_{i1}(s)| K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{v_2}\setminus\mathcal{J}_{v_1}} \bigcap_{k\in\mathcal{J}_{v_1}} (\gamma_{k1}(s), \gamma_{j1}(s)]\right)$$

which is clearly positive. □

Lemma 9. *Let C be the copula of $\mathcal{U}(0,1)$-random variables X, Y such that $\alpha(X) = \beta(Y)$ a.s. for some $\alpha, \beta \in \mathscr{T}_P$. If A_t, $t \in [0,1]$, are copulas defined in Lemma 8, then for $x, y \in \mathbb{I}$, $A_t\left(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\beta}(y,t)\right)$ is measurable in $t \in [0,1]$.*

Proof. For $s \in I_1$, let $u(s) := \partial_2 C_{e,\alpha}(x, \alpha(s))$ and $v(s) := \partial_2 C_{e,\beta}(y, \alpha(s))$, whose values are in $P(s)$ and $Q(s)$, respectively. By Remark 4(3), $u(s)$ and $v(s)$ are Borel functions of s. For each $\eta \in Q(s)$, since $\gamma_{j1}(s) < d_j$, Lemma 6(4) yields $K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_\eta}(0, \gamma_{j1}(s)]\right) = K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_\eta}(0, d_j]\right)$, which is measurable in s because $s_{(i)}$ is a Borel function of s. So $A_{\alpha(s)}(\mu, \eta)$ is measurable in s for $(\mu, \eta) \in P(s) \times Q(s)$, and hence for $(u, v) \in [0, 1]^2$ as $A_{\alpha(s)}(u, v)$ is a bilinear interpolation of $A_{\alpha(s)}(\mu, \eta)$'s. Consequently, the composition $A_{\alpha(s)}\left(\partial_2 C_{e,\alpha}\left(x, \alpha(s)\right), \partial_2 C_{e,\beta}\left(y, \alpha(s)\right)\right)$ is measurable in s. Using a change of variable $t = \alpha(s)$, the proof is complete. □

Theorem 10. *Let random variables $X, Y \sim \mathcal{U}(0, 1)$ be such that $\alpha(X) = \beta(Y)$ a.s. for some $\alpha, \beta \in \mathscr{T}_P$. Then there exists $\mathcal{A} = \{A_t\}_{t\in[0,1]} \subseteq \mathscr{C}$ such that $C_{(X,Y)} = C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}$.*

Proof. Putting $\mathcal{A} := \{A_t\}_{t\in[0,1]}$ as defined in Lemma 8, we have, using Lemma 9, that

$$\begin{aligned} C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}(x, y) &= \int_0^1 A_t(\partial_2 C_{e,\alpha}(x, t), \partial_2 C_{e,\beta}(y, t))\, dt \\ &= \int_{a_1}^{b_1} |\alpha'(s)| A_{\alpha(s)}\left(u(s), v(s)\right)\, ds, \end{aligned} \tag{4.4}$$

where $u(s) := \partial_2 C_{e,\alpha}\left(x, \alpha(s)\right)$, $v(s) := \partial_2 C_{e,\beta}(y, \alpha(s))$ and the last equality uses the change of variable $t = \alpha(s)$ for $s \in I_1$. Denote $C := C_{(X,Y)}$ and K_C its Markov kernel. The rest of the proof is devoted to deriving that (4.4) equals $\int_0^x K_C\left(s, [0, y]\right) ds$ which, by equation (2.1), is equal to $C(x, y)$. By Remark 4(3), if $(x, y) \in I_p \times J_q$ then, with $\{\hat{\mu}_p, \check{\mu}_p\} = \left\{\mu_p, \mu_p^*\right\}$ and $\{\hat{\eta}_q, \check{\eta}_q\} = \left\{\eta_q, \eta_q^*\right\}$, for a.e. $s \in I_1$,

$$u(s) = \hat{\mu}_p \mathbb{1}_{(a_1, x_{(1)})}(s) + \check{\mu}_p \mathbb{1}_{(x_{(1)}, b_1)}(s) \quad \text{and} \quad v(s) = \hat{\eta}_q \mathbb{1}_{(a_1, \gamma_{q1}^{-1}(y))}(s) + \check{\eta}_q \mathbb{1}_{(\gamma_{q1}^{-1}(y), b_1)}(s),$$

where $\hat{\mu}_p := \begin{cases} \mu_p & \text{if } \alpha_{p1} \text{ is increasing;} \\ \mu_p^* & \text{if } \alpha_{p1} \text{ is decreasing,} \end{cases}$ and $\hat{\eta}_q := \begin{cases} \eta_q & \text{if } \gamma_{q1} \text{ is increasing;} \\ \eta_q^* & \text{if } \gamma_{q1} \text{ is decreasing.} \end{cases}$

Hence, by Lemma 9, $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}(x, y)$ becomes

$$\int_{a_1}^{x_{(1)}} |\alpha'(s)| A_{\alpha(s)}\left(\hat{\mu}_p, \hat{\eta}_q\right) \mathrm{d}s + \int_{x_{(1)}}^{\gamma_{q1}^{-1}(y)} |\alpha'(s)| A_{\alpha(s)}\left(\check{\mu}_p, \hat{\eta}_q\right) \mathrm{d}s + \int_{\gamma_{q1}^{-1}(y)}^{b_1} |\alpha'(s)| A_{\alpha(s)}\left(\check{\mu}_p, \check{\eta}_q\right) \mathrm{d}s \tag{A}$$

if $x_{(1)} \leq \gamma_{q1}^{-1}(y)$ and

$$\int_{a_1}^{\gamma_{q1}^{-1}(y)} |\alpha'(s)| A_{\alpha(s)}\left(\hat{\mu}_p, \hat{\eta}_q\right) \mathrm{d}s + \int_{\gamma_{q1}^{-1}(y)}^{x_{(1)}} |\alpha'(s)| A_{\alpha(s)}\left(\hat{\mu}_p, \check{\eta}_q\right) \mathrm{d}s + \int_{x_{(1)}}^{b_1} |\alpha'(s)| A_{\alpha(s)}\left(\check{\mu}_p, \check{\eta}_q\right) \mathrm{d}s \tag{B}$$

if $x_{(1)} > \gamma_{q1}^{-1}(y)$.

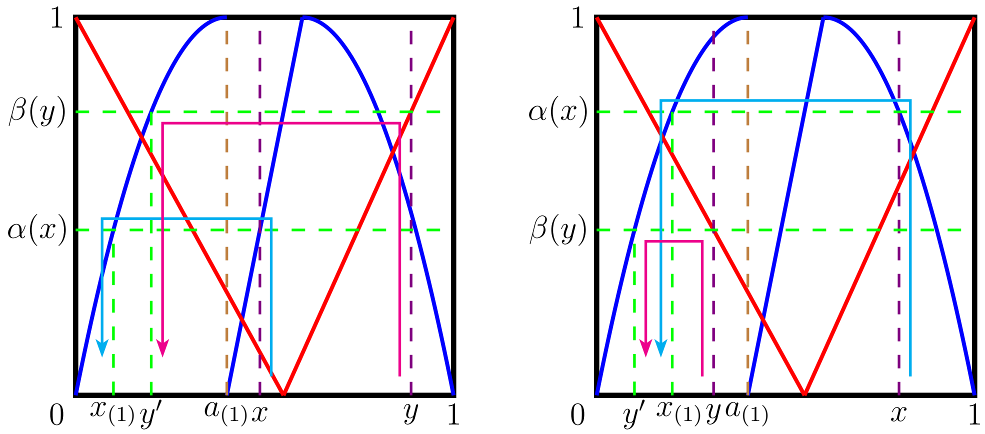

Figure 2: x, y considered in Case 1 (left) and Case 2 (right) where $y' = \gamma_{q1}^{-1}(y)$.

Case 1 $x_{(1)} \leq \gamma_{q1}^{-1}(y)$: By (A), $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}(x, y)$ equals

$$
\begin{aligned}
&\int_{a_1}^{x_{(1)}} \sum_{i \in \mathcal{I}_{\hat\mu_p}} |\alpha'_{i1}(s)| K_C \left(s_{(i)}, \bigcup_{j \in \mathcal{J}_{\hat\eta_q}} [0, \gamma_{j1}(s)] \right) ds \\
&\quad + \int_{x_{(1)}}^{\gamma_{q1}^{-1}(y)} \sum_{i \in \mathcal{I}_{\check\mu_p}} |\alpha'_{i1}(s)| K_C \left(s_{(i)}, \bigcup_{j \in \mathcal{J}_{\hat\eta_q}} [0, \gamma_{j1}(s)] \right) ds \\
&\quad + \int_{\gamma_{q1}^{-1}(y)}^{b_1} \sum_{i \in \mathcal{I}_{\check\mu_p}} |\alpha'_{i1}(s)| K_C \left(s_{(i)}, \bigcup_{j \in \mathcal{J}_{\check\eta_q}} [0, \gamma_{j1}(s)] \right) ds \\
&= \sum_{i \in \mathcal{I}_{\hat\mu_p}} \int_{a_1}^{x_{(1)}} |\alpha'_{i1}(s)| K_C(s_{(i)}, [0, y]) ds + \sum_{i \in \mathcal{I}_{\check\mu_p}} \int_{x_{(1)}}^{\gamma_{q1}^{-1}(y)} |\alpha'_{i1}(s)| K_C(s_{(i)}, [0, y]) ds \\
&\quad + \sum_{i \in \mathcal{I}_{\check\mu_p}} \int_{\gamma_{q1}^{-1}(y)}^{b_1} |\alpha'_{i1}(s)| K_C(s_{(i)}, [0, y]) ds \\
&= \sum_{\substack{i \in \mathcal{I}_{\hat\mu_p} \\ \alpha_{i1} \text{ inc}}} \int_{a_i}^{x_{(i)}} K_C(s, [0, y]) ds + \sum_{\substack{i \in \mathcal{I}_{\check\mu_p} \\ \alpha_{i1} \text{ inc}}} \int_{x_{(i)}}^{\gamma_{qi}^{-1}(y)} K_C(s, [0, y]) ds + \sum_{\substack{i \in \mathcal{I}_{\check\mu_p} \\ \alpha_{i1} \text{ inc}}} \int_{\gamma_{qi}^{-1}(y)}^{b_i} K_C(s, [0, y]) ds \\
&\quad + \sum_{\substack{i \in \mathcal{I}_{\hat\mu_p} \\ \alpha_{i1} \text{ dec}}} \int_{x_{(i)}}^{b_i} K_C(s, [0, y]) ds + \sum_{\substack{i \in \mathcal{I}_{\check\mu_p} \\ \alpha_{i1} \text{ dec}}} \int_{\gamma_{qi}^{-1}(y)}^{x_{(i)}} K_C(s, [0, y]) ds + \sum_{\substack{i \in \mathcal{I}_{\check\mu_p} \\ \alpha_{i1} \text{ dec}}} \int_{a_i}^{\gamma_{qi}^{-1}(y)} K_C(s, [0, y]) ds \\
&= \sum_{\substack{i \in \mathcal{I}_{\mu_p^*} \\ \alpha_{i1} \text{ inc}}} \int_{a_i}^{b_i} K_C\left(s, [0, y]\right) \mathrm{d}s + \sum_{\substack{i \in \mathcal{I}_{\mu_p^*} \\ \alpha_{i1} \text{ dec}}} \int_{a_i}^{b_i} K_C\left(s, [0, y]\right) \mathrm{d}s + \int_{a_p}^{x} K_C\left(s, [0, y]\right) \mathrm{d}s \\
&= \int_0^x K_C(s, [0, y]) ds.
\end{aligned}
$$

The second equality holds because of the following reasons.

- In the case γ_{q1} is increasing, we use $\gamma_{qi} = \gamma_{q1} \circ \alpha_{1i}$ and Remark 4(2) to show that for $s \in (a_1, \gamma_{q1}^{-1}(y))$, $s_{(i)} = \alpha_{i1}(s) < \alpha_{i1}\gamma_{q1}^{-1}(y) = \gamma_{qi}^{-1}(y)$ or $s_{(i)} > \gamma_{qi}^{-1}(y)$ if α_{i1} is increasing or decreasing respectively. Both situations lead to $\gamma_{qi}(s_{(i)}) < y$. Thus by Lemma 6(4),

$$K_C\left(s_{(i)}, [0,y]\right) = K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\eta_q}} [0, \gamma_{ji}(s_{(i)})]\right) = K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\hat{\eta}_q}} [0, \gamma_{j1}(s)]\right).$$

- Likewise, for $s \in (\gamma_{q1}^{-1}(y), b_1)$, we have

$$K_C\left(s_{(i)}, [0,y]\right) = K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\eta_q^*}} [0, \gamma_{j1}(s)]\right) = K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\check{\eta}_q}} [0, \gamma_{j1}(s)]\right).$$

- We can verify in the similar way for the case γ_{q1} is decreasing.

Note the changes of variable $s_{(i)} = \alpha_{i1}(s)$ for $i \in \mathcal{I}_{\mu_p}$ performed in the third equality. While the next to last one follows from the assumption of α_{p1}.

Case 2 $x_{(1)} > \gamma_{q1}^{-1}(y)$: By (B), $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}(x,y)$ equals

$$\begin{aligned}
&\int_{a_1}^{\gamma_{q1}^{-1}(y)} \sum_{i\in\mathcal{I}_{\hat{\mu}_p}} |\alpha_{i1}'(s)| K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\hat{\eta}_q}} [0, \gamma_{j1}(s)]\right) ds \\
&+ \int_{\gamma_{q1}^{-1}(y)}^{x_{(1)}} \sum_{i\in\mathcal{I}_{\hat{\mu}_p}} |\alpha_{i1}'(s)| K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\check{\eta}_q}} [0, \gamma_{j1}(s)]\right) ds \\
&+ \int_{x_{(1)}}^{b_1} \sum_{i\in\mathcal{I}_{\check{\mu}_p}} |\alpha_{i1}'(s)| K_C\left(s_{(i)}, \bigcup_{j\in\mathcal{J}_{\check{\eta}_q}} [0, \gamma_{j1}(s)]\right) ds := T_1 + T_2 + T_3.
\end{aligned}$$

By a similar argument as used in the previous subcase, we have

$$\begin{aligned}
T_1 &= \sum_{\substack{i\in\mathcal{I}_{\hat{\mu}_p}\\ \alpha_{i1}\text{ inc}}} \int_{a_i}^{\gamma_{qi}^{-1}(y)} K_C\left(s,[0,y]\right) \mathrm{d}s + \sum_{\substack{i\in\mathcal{I}_{\hat{\mu}_p}\\ \alpha_{i1}\text{ dec}}} \int_{\gamma_{qi}^{-1}(y)}^{b_i} K_C\left(s,[0,y]\right) \mathrm{d}s, \\
T_2 &= \sum_{\substack{i\in\mathcal{I}_{\hat{\mu}_p}\\ \alpha_{i1}\text{ inc}}} \int_{\gamma_{qi}^{-1}(y)}^{x_{(i)}} K_C\left(s,[0,y]\right) \mathrm{d}s + \sum_{\substack{i\in\mathcal{I}_{\hat{\mu}_p}\\ \alpha_{i1}\text{ dec}}} \int_{x_{(i)}}^{\gamma_{qi}^{-1}(y)} K_C\left(s,[0,y]\right) \mathrm{d}s, \text{ and} \\
T_3 &= \sum_{\substack{i\in\mathcal{I}_{\check{\mu}_p}\\ \alpha_{i1}\text{ inc}}} \int_{x_{(i)}}^{b_i} K_C\left(s,[0,y]\right) \mathrm{d}s + \sum_{\substack{i\in\mathcal{I}_{\check{\mu}_p}\\ \alpha_{i1}\text{ dec}}} \int_{a_i}^{x_{(i)}} K_C\left(s,[0,y]\right) \mathrm{d}s.
\end{aligned}$$

Thus, $C_{e,\alpha} *_{\mathcal{A}} C_{\beta,e}(x,y) = \int_0^x K_C\left(s,[0,y]\right) \mathrm{d}s$ as desired. Note that the proof above still holds in the case that $\mathcal{I}_{\mu_p^*} = \varnothing$. So we finish the proof. □

5 Examples

Example 11. *Let $T = [t_{ij}]_{m\times n}$ be a transformation matrix, i.e., i) all entries are nonnegative; ii) the sum of all entries is one; iii) every row and column has at least one nonzero entry, with partitions $P = \left\{ a_k := \sum_{i=1}^{k}\sum_{j=1}^{n} t_{ij} \right\}$ on the x-axis and $Q = \left\{ b_\ell := \sum_{j=1}^{\ell}\sum_{i=1}^{m} t_{ij} \right\}$ on the y-axis. Then the checkmin copula generated by T is given by*

$$C_T(x, y) = \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} M\left(F_i(x), G_j(y)\right)$$

where $F_i(x) = \min\left\{ \frac{x - a_{i-1}}{a_i - a_{i-1}}, 1 \right\} \mathbb{1}_{(a_{i-1},1]}(x)$ and $G_j(y) = \min\left\{ \frac{y - b_{j-1}}{b_j - b_{j-1}}, 1 \right\} \mathbb{1}_{(b_{j-1},1]}(y)$. Note that on $I_i \times J_j := (a_{i-1}, a_i) \times (b_{j-1}, b_j)$ for which $t_{ij} \neq 0$, we have $\frac{x - a_{i-1}}{a_i - a_{i-1}} = \frac{y - b_{j-1}}{b_j - b_{j-1}}$ a.s. Hence

$$\alpha(x) = \sum_{i=1}^{m} \frac{x - a_{i-1}}{a_i - a_{i-1}} \mathbb{1}_{I_i}(x) \quad \text{and} \quad \beta(y) = \sum_{j=1}^{n} \frac{y - b_{j-1}}{b_j - b_{j-1}} \mathbb{1}_{J_j}(y).$$

Moreover, for each $s \in (0, a_1)$,

- $\mu_k = \sum_{i=1}^{k} \frac{1}{|\alpha'(s_{(i)})|} = \sum_{i=1}^{k} (a_i - a_{i-1}) = a_k$ *for all $k = 1, 2, \ldots, m$; and*
- $\eta_\ell = \sum_{j=1}^{\ell} \frac{1}{|\beta'(t^{(j)})|} = \sum_{j=1}^{\ell} (b_j - b_{j-1}) = b_\ell$, *where $t = \frac{b_1}{a_1} s$, for all $\ell = 1, 2, \ldots, n$.*
- *Note that $K_{C_T}(u, [0, v]) = \partial_1 C_T(u, v)$ for a.e. $(u, v) \in \mathbb{I}^2$. Thus, by Lemma 6(3,4), for any $k \in \{1, 2, \ldots, m\}$ and $\ell \in \{1, 2, \ldots, n\}$,*

$$\begin{aligned}
K_{C_T}\left(s_{(k)}, [0, \gamma_{\ell 1}(s)]\right) &= K_{C_T}\left(s_{(k)}, [0, b_\ell]\right) = \partial_1 C_T\left(s_{(k)}, b_\ell\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} \partial_{s_{(k)}} M\left(F_i(s_{(k)}), G_j(b_\ell)\right) \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{\ell} t_{ij} \partial_{s_{(k)}} M(1, 1) + \sum_{j=1}^{\ell} t_{kj} \partial_{s_{(k)}} M\left(\frac{s_{(k)} - a_{k-1}}{a_k - a_{k-1}}, 1 \right) \\
&= \frac{1}{a_k - a_{k-1}} \sum_{j=1}^{\ell} t_{kj}.
\end{aligned}$$

From information above, we have the subcopula $A_{\alpha(s)}$ is given by

$$A_{\alpha(s)}\left(a_k, b_\ell\right) = \frac{1}{|\alpha'(s)|} \sum_{i=1}^{k} |\alpha'_{i1}(s)| K_{C_T}\left(s_{(i)}, [0, \gamma_{\ell 1}(s)]\right)$$

$$= a_1 \sum_{i=1}^{k} \left(\frac{a_i - a_{i-1}}{a_1}\right)\left(\frac{1}{a_i - a_{i-1}} \sum_{j=1}^{\ell} t_{ij}\right) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} t_{ij}.$$

Note that both the domain and the function values of $A_{\alpha(s)}$ are independent of s. Therefore, all $A_{\alpha(s)}$ can be chosen to be the same copula A whose value at (a_k, b_ℓ) is $\sum_{i=1}^{k}\sum_{j=1}^{\ell} t_{ij}$, and so $C_T = C_{e,\alpha} *_A C_{\beta,e}$.

The figure below shows the support of $C_T, C_{e,\alpha}$ and $C_{\beta,e}$ where $T = \begin{bmatrix} 0.2 & 0.05 & 0.05 & 0 \\ 0 & 0.25 & 0.15 & 0 \\ 0.05 & 0 & 0.05 & 0.2 \end{bmatrix}$.

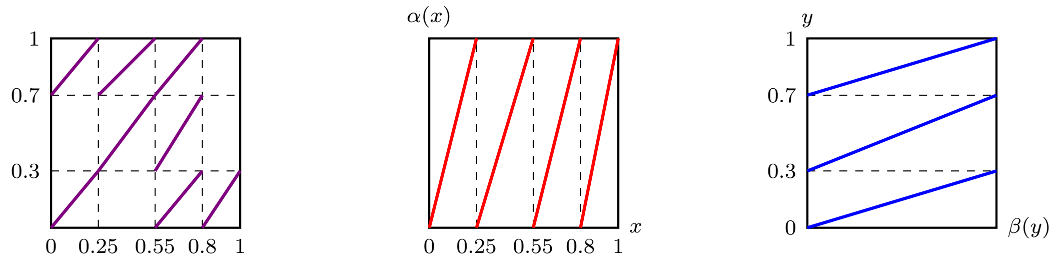

Figure 3: Support of copulas $C_T, C_{e,\alpha}$ and $C_{\beta,e}$, respectively.

Example 12. *Let $f(x) = 1 - \sqrt{1 - x^2}$ where $x \in \mathbb{I}$ and $C(x, y) = \min\left\{x, y, \frac{1}{2}\left(x^2 + y^2\right)\right\}$, for $(x, y) \in \mathbb{I}^2$, be a copula with $\operatorname{supp}(C) = G_f \cup G_{f^{-1}}$ where G_h denote the graph of h.*

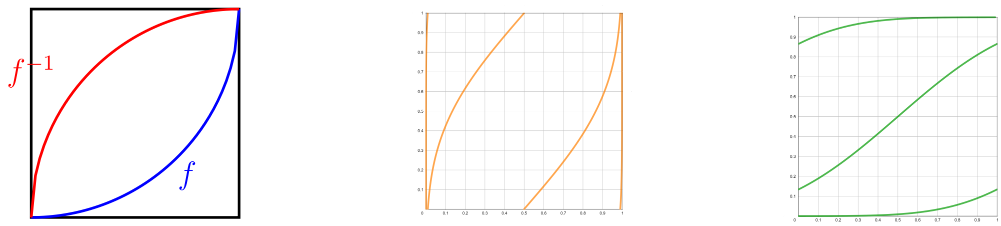

Figure 4: support of $C, C_{e,\alpha}$ and $C_{\beta,e}$ respectively.

By [14], we have for any $n \in \mathbb{Z}$,

$$\forall x \in I_n := \left[f^{2n+2}\left(\frac{1}{2}\right), f^{2n}\left(\frac{1}{2}\right)\right), \quad \alpha(x) = \gamma\left(f^{-2n}(x)\right) \quad \text{and}$$
$$\forall y \in J_n := \left[f^{2n+1}\left(\frac{1}{2}\right), f^{2n-1}\left(\frac{1}{2}\right)\right), \quad \beta(y) = \gamma\left(f^{-2n+1}(y)\right),$$

where $\gamma(t) = \lambda\left(\dot{\bigcup_{n\in\mathbb{Z}}} f^{2n}\left[f^2\left(\frac{1}{2}\right), t\right)\right)$ for $t \in \left[f^2\left(\frac{1}{2}\right), \frac{1}{2}\right]$ as in the figure 4. Note that for any $s \in I_0$ and $t \in J_0$ with $\alpha(s) = \beta(t)$, i.e. $s = f(t)$, $\mu_n^* = \mu_{n+1}$ and $\eta_n^* = \eta_{n+1}$ for all $n \in \mathbb{Z}$. Moreover, for any $s \in I_0$,

- $\mu_k = \sum_{I_i \preceq I_k} \frac{1}{\left|\alpha'\left(s_{(i)}\right)\right|} = \sum_{i=k}^{\infty} \frac{1}{\alpha'\left(f^{2i}(s)\right)} = \frac{1}{\gamma'(s)} \sum_{i=k}^{\infty} \frac{1}{\left(f^{-2i}\right)'\left(f^{2i}(s)\right)}$; *and*

- $\eta_\ell = \sum_{J_j \preceq J_\ell} \frac{1}{|\beta'(t^{(j)})|} = \sum_{j=\ell}^{\infty} \frac{1}{\beta'(f^{2j}(t))} = \frac{1}{\gamma'(s)} \sum_{j=\ell}^{\infty} \frac{1}{(f^{-2j+1})'(f^{2j}(t))}.$
- *By Lemma 6(3,4), for any $k, \ell \in \mathbb{Z}$,*

$$
\begin{aligned}
K_C\left(s_{(k)}, \bigcup_{J_j \preceq J_\ell} [0, \gamma_{j0}(s)]\right) &= K_C\left(s_{(k)}, [0, \gamma_{\ell 0}(s)]\right) = K_C\left(s_{(k)}, \left[0, f^{2\ell-1}\left(\frac{1}{2}\right)\right]\right) \\
&= \partial_1 C\left(s_{(k)}, f^{2\ell-1}\left(\frac{1}{2}\right)\right) \\
&= \begin{cases} 1 & \text{if } s_{(k)} < f^{2\ell}\left(\frac{1}{2}\right); \\ s_{(k)} & \text{if } f^{2\ell}\left(\frac{1}{2}\right) < s_{(k)} < f^{2\ell-2}\left(\frac{1}{2}\right); \\ 0 & \text{if } s_{(k)} > f^{2\ell-2}\left(\frac{1}{2}\right). \end{cases} \\
&= \begin{cases} 1 & \text{if } k \geq \ell; \\ s_{(\ell-1)} & \text{if } k = \ell - 1; \\ 0 & \text{if } k < \ell - 1. \end{cases}
\end{aligned}
$$

From information above, we have the subcopula $A_{\alpha(s)}$ is given by

$$
\begin{aligned}
A_{\alpha(s)}(\mu_k, \eta_\ell) &= \frac{1}{|\alpha'(s)|} \sum_{I_i \preceq I_k} |\alpha'_{i0}(s)| K_C\left(s_{(i)}, \bigcup_{J_j \preceq J_\ell} [0, \gamma_{j0}(s)]\right) \\
&= \begin{cases} \sum_{i=k}^{\ell-2} \frac{1}{\alpha'(s_{(i)})} \cdot 0 + \frac{s_{(\ell-1)}}{\alpha'(s_{(\ell-1)})} + \sum_{i=\ell}^{\infty} \frac{1}{\alpha'(s_{(i)})} & \text{if } k < \ell - 1; \\ \frac{s_{(\ell-1)}}{\alpha'(s_{(\ell-1)})} + \sum_{i=\ell}^{\infty} \frac{1}{\alpha'(s_{(i)})} & \text{if } k = \ell - 1; \\ \sum_{i=k}^{\infty} \frac{1}{\alpha'(s_{(i)})} & \text{if } k \geq \ell, \end{cases} \\
&= \begin{cases} \mu_\ell + \frac{f^{2\ell-2}(s)}{\alpha'(f^{2\ell-2}(s))} & \text{if } k < \ell; \\ \mu_k & \text{if } k \geq \ell, \end{cases}
\end{aligned}
$$

which can be represented by the infinite matrix A_s below where $a_{k\ell} = A_{\alpha(s)}(\mu_k, \eta_\ell)$ for all $k, \ell \in \mathbb{Z}$.

$$
\begin{array}{ccccccc} & & & \leftarrow k & & & \\ \cdots & 2 & 1 & 0 & -1 & -2 & \cdots \end{array}
$$

$$
A_s = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \mu_2 & \mu_1 & \mu_0 & \mu_{-1} & \mu_{-2} & \cdots \\
\cdots & \mu_2 & \mu_1 & \mu_0 & \mu_{-1} & \mu_{-1} + \frac{f^{-4}(s)}{\alpha'(f^{-4}(s))} & \cdots \\
\cdots & \mu_2 & \mu_1 & \mu_0 & \mu_0 + \frac{f^{-2}(s)}{\alpha'(f^{-2}(s))} & \mu_0 + \frac{f^{-2}(s)}{\alpha'(f^{-2}(s))} & \cdots \\
\cdots & \mu_2 & \mu_1 & \mu_1 + \frac{s}{\alpha'(s)} & \mu_1 + \frac{s}{\alpha'(s)} & \mu_1 + \frac{s}{\alpha'(s)} & \cdots \\
\cdots & \mu_2 & \mu_2 + \frac{f^2(s)}{\alpha'(f^2(s))} & \mu_2 + \frac{f^2(s)}{\alpha'(f^2(s))} & \mu_2 + \frac{f^2(s)}{\alpha'(f^2(s))} & \mu_2 + \frac{f^2(s)}{\alpha'(f^2(s))} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{array}{c} \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{array}
\quad \downarrow \ell
$$

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